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**MODELING THE DEMAND FOR SPARE PARTS:
ESTIMATING THE VARIANCE-TO-MEAN RATIO
AND OTHER ISSUES**

James S. Hodges

May 1985

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Prepared for

The United States Air Force

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Mathematical models are commonly used to study the performance of the Air Force's spare parts supply and repair systems. But accurate evaluations of supply policies are not possible without accurate models of the supply system, and models that understate the variability in the supply system will bias evaluations in favor of policies that rely on accurate predictions of part failures. This Note examines the model for part failures used in The Rand Corporation Supply System model, Dyna-METRIC. The ability to predict levels of parts failures is strongly affected by at least two types of uncertainty: about the numbers of failures that will occur assuming the model is correct, and about the adequacy of the model as an approximation of the process generating parts failures. The author suggests that a model that allows more variability, such as a negative binomial model, would be more appropriate for dealing with the first type of uncertainty, and the second type can be accommodated partly by using models with more parameters.

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ESTIMATING THE VARIANCE-TO-MEAN RATIO
AND OTHER ISSUES

James S. Hodges

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Rand

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PREFACE

This Note covers work done on the problem of modeling and estimating the demand for aircraft spare parts while the author was a consultant to The Rand Corporation in the summer of 1983. The research began under the Project AIR FORCE Resource Management Program study entitled "The Driving Inputs and Assumptions of Stockage Assessment Models." The Note is being published as part of a follow-on study entitled "Enhancing the Integration and Responsiveness of the Support System to Meet Wartime and Peacetime Uncertainties."

This work grew out of a study of parts failure data from several Air Force units, strongly indicating that either current models of part failure behavior or prevailing beliefs about the inherent stability of this behavior, or both, were wrong. The results here support this indication, but they are of general interest for modeling and estimating nonhomogeneous Poisson processes.

This Note will be of interest to those concerned with forecasting inventory requirements in the Department of Defense or industry.

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SUMMARY

Mathematical models are commonly used to study the performance of the Air Force's spare parts supply and repair systems. But accurate evaluations of supply policies are not possible without accurate models of the supply system, and models that understate the variability in the supply system will bias evaluations in favor of policies that rely on accurate predictions of part failures. This Note examines the model for part failures used in The Rand Corporation Supply System model Dyna-METRIC.

Section I gives a short description of Dyna-METRIC and then a longer description of common probability models for part failures. The strengths and weaknesses of these models are discussed, with particular attention to sources of variability in observed part failure behavior that the models do not appear to capture.

Section II has two purposes. The first is to examine the plausibility of Dyna-METRIC's current probability model for part failures in the light of some new Air Force data. This model treats the number of failures of a particular part, at a particular air base, in a time period of given length, as a Poisson random variable with mean λf , where f is the total number of hours flown in that time period, and λ is an unknown constant characteristic of the part and airbase. The second purpose is to derive some new properties of V , a commonly used estimator of the variance-to-mean ratio, under three different probability models for part failures. These properties of V indicate strongly that Dyna-METRIC's current probability model does not permit enough variability to credibly explain the Air Force data. Further, although the data do indicate that it is preferable to model mean part failures as $\alpha + \beta f$ for α and β unknown constants and α not necessarily zero, a Poisson model with this mean still does not allow enough variability. A negative binomial model with mean $\alpha + \beta f$ is preferred.

Finally, Sec. II shows that V is always biased low for the probability model in which it is intended to be used--i.e., with part failures distributed as negative binomial random variables with mean λf

and variance-to-mean ratio p . This bias is an increasing function of p for a fixed number of total expected failures and can be very large for large p .

Section III contains suggestions for estimating the parameters of the models recommended in Sec. II. Maximum likelihood estimates are suggested because they are tractable and because they appear to solve the bias problem noted above.

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I. THE REAL WORLD PROBLEM, SOME CURRENT MODELS, AND SOME MODELING ISSUES

INTRODUCTION

One of the missions assigned to the Air Force is to be able to relocate fighting units and associated service and supply facilities to far-flung places on short notice, to meet a variety of policy commitments. A remarkable amount of materiel is needed to keep such units operational, and airlifting capacity is limited and expensive. Because of this limited capacity, several policy choices must be made. One such choice is how best to handle spare parts supply requirements-- i.e., is it possible to design spare parts kits that don't take up too much airlift space but are sufficient to maintain an adequate number of operational planes for some desired length of time; or would it be preferable to have a "responsive" supply system in which, for example, a dedicated fleet of planes hauls spare parts around as needed, reducing spare parts kits airlifted with the units accordingly? Given the choice of a general approach to supplying parts, how often should shipments be made, how large should these shipments be? And so on.

These kinds of questions have long been examined with the help of mathematical models intended to simulate the wartime performance of the spare parts supply and repair system. One model developed at Rand and in wide use within the Air Force is called Dyna-METRIC (for Dynamic Multi-Echelon Technique for Recoverable Item Control); henceforth I will often refer specifically to Dyna-METRIC, but its features are not atypical of such models.

These models attempt to characterize the (1) *numbers of failures of parts on planes*, where proneness to failure is allowed to depend on the intensity of use as dictated by a war scenario (thus "Dynamic"); (2) *repair times* of those parts that can be repaired, either at the forward airbase or at some more centralized rearward repair facility (thus "multi-echelon"); (3) *transportation times and other delays*, where relevant; and (4) *stocking requirements*, the levels of stocks of parts kept at the forward base and at the more centralized repair facility, and levels of and timing of supplies of new parts.

The general scheme of the Dyna-METRIC model is depicted in Fig. 1, which is a simplified version of Fig. 1 in Hillestad (1982). In Dyna-METRIC, the numbers of part failures are modeled as random variables whose probability distributions depend on the number of hours flown, the measure of intensity of use. To be more specific, consider a particular Air Force unit, and a part i , which is in all of the planes in that unit. Let $X_j(i)$ denote the number of failures on part i in that unit in time period j . In Dyna-METRIC, the parameters of the probability distribution of $X_j(i)$ are assumed to depend on total hours flown by the unit in the j th time period. The effects of different war scenarios on failures are examined by manipulating the number of hours flown, the number of planes used, and other factors affecting the distributions of $X_j(i)$ for the various parts.

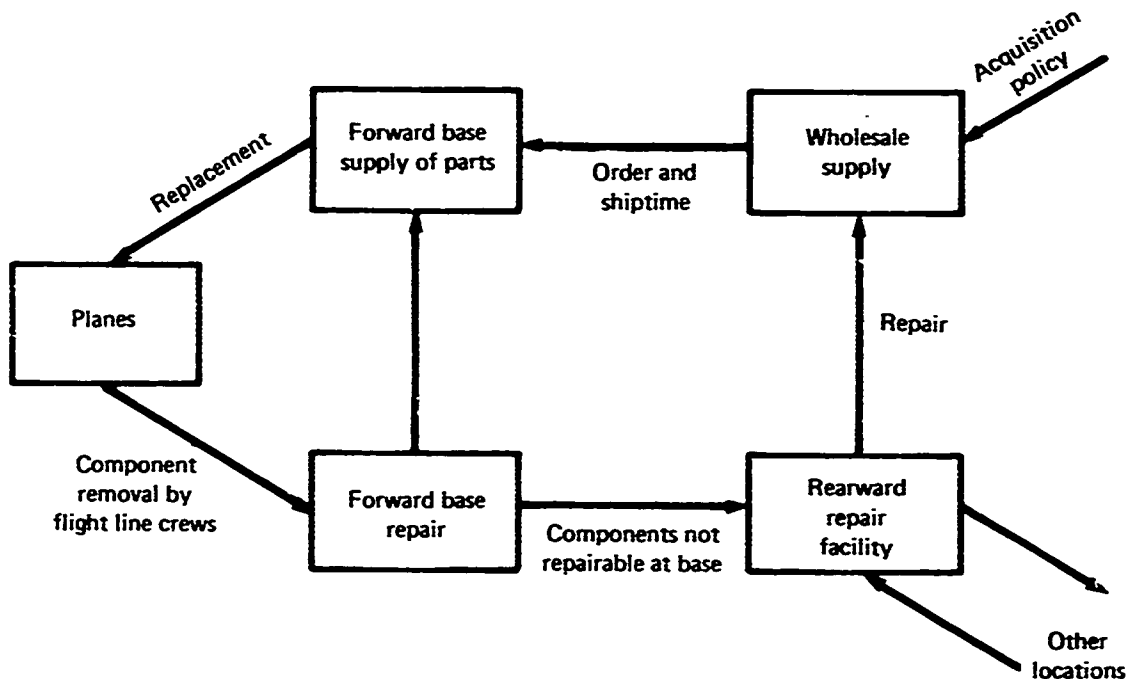


Fig. 1 – Simplified scheme of parts repair and supply model

Repair times for individual failed parts are treated as random variables having exponential or degenerate distributions, and assumptions are made about the probabilities that parts can be repaired only at the rearward centralized facility. The exponential distribution was chosen for mathematical convenience, but it seems to be a satisfactory assumption. Both under the assumption of an infinite number of repair stands (Crawford, 1981) and, as far as is known, under the assumption of a finite number of repair stands, the supply system performance measurements produced by Dyna-METRIC are fairly insensitive to the distributional forms assumed for repair times.

The effects of various stockage policies can be introduced by varying purchases from vendors and decisions about allocating spare parts to the stock of the forward bases or the rearward facility. It is assumed that stocking policy is nonstochastic, and these policies are represented by changes at prescribed times in the numbers in stock at the bases and the rearward facility.

The purpose of these models is to connect dollars and cents policy decisions to aircraft performance of assigned missions. Dyna-METRIC and other models produce several performance measures. Some are inventory measures, such as the probability distribution of the maximum backorder across all parts, the probability that a failure of a particular part will be filled immediately from stock, and so on. These measures are derived analytically, based on the distributions assumed for failures and repair times. More relevant to the policy questions, if mission requirements can be specified in terms of numbers of planes with particular capabilities, it is possible to derive a probability distribution for mission capability from the already noted inventory measures.

MODELING FAILURES (DEMANDS)

These performance measures are no better than the model that generates them. The particular area of concern here is modeling the numbers of failures, or demands, and estimating the parameters in those models.

A common assumption is that the number of failures X in a period of given length has a Poisson distribution with parameter μ , in which case

$$P(X = k) = e^{-\mu} \mu^k / k! \quad \text{for } k = 0, 1, 2, \dots$$

Under this assumption, both the mean and variance of X are equal to μ , so that the variance-to-mean ratio $\rho = \text{var}(X)/E(X)$ is equal to 1. To model applications in which ρ exceeds one, a negative binomial distribution is often hypothesized. Here,

$$P(X = k) = \binom{k + r - 1}{k} (1 - p)^k p^r \quad \text{for } k = 0, 1, 2, \dots,$$

where $r > 0$, $0 < p < 1$. In this case, $E(X) = r(1 - p)/p$ and $\text{var}(X) = r(1 - p)/p^2$, so that $\rho = 1/p > 1$. As is well known, the Poisson distribution is a limit of negative binomial distributions as $p \rightarrow 1$ and $r \rightarrow \infty$ in such a way that $r(1 - p)$ remains fixed at μ (Feller, 1968, p. 172).

The current version of Dyna-METRIC assumes that demand for a particular part for a particular Air Force unit in some period of given length is either Poisson or negative binomial. When the Poisson distribution is used, its mean is assumed to equal λf , where λ is an unknown constant peculiar to the part and unit, and f is the number of hours flown by the unit in the given period. When the negative binomial distribution is used, its mean is also assumed to be λf , and its variance-to-mean ratio is assumed to be ρ , so that $r = \lambda f/(\rho - 1)$ and $p = 1/\rho$.

The negative binomial is a "compound Poisson" distribution in both senses in which that term is commonly used. In the first sense, usually associated with probabilists, the random variable $S_N = X_1 + X_2 + \dots + X_N$ can be shown to be negative binomial if the X_j are independently and identically distributed with

$$P(X_j = n) = (1 - p)^n / (-n \ln p), \quad n = 1, 2, \dots,$$

for $0 < p < 1$, and N is independently distributed as a Poisson random

variable with $\mu = -r \ln p$ (Feller, 1968, pp. 268-269 and 286-291). In the second sense, usually associated with statisticians, if the conditional distribution of a random variable X , given its mean μ , is Poisson, and μ itself has a gamma distribution with density

$$g(\mu) = \beta^{-\alpha} \mu^{\alpha-1} e^{-\mu/\beta} / \Gamma(\alpha),$$

then unconditionally X has a negative binomial distribution with $r = \alpha$ and $p = (\beta + 1)^{-1}$ (Hogg and Craig, 1970, pp. 99-211). More generally, compound Poisson distributions are defined by replacing $P(X_j = n)$ with an appropriate distribution in the first sense, or by replacing the gamma distribution appropriately in the second sense.

Aside from mathematical tractability, the use of compound Poisson models has been rationalized by at least two substantive considerations. First, it follows from Levy's work on infinitely divisible distributions that any arrival process where the numbers of arrivals in disjoint time intervals are independent is Poisson or compound Poisson in the first sense above (Feller, 1968, pp. 289-290; Crawford, 1981, p. 10). Thus an assumption of independence of failures in disjoint time intervals justifies using compound Poisson models, although it does not justify the use of any particular compound Poisson model. Second, under some conditions the Poisson distribution is a good approximation to the distribution of the sum $S_n = X_1 + \dots + X_n$ of n mutually independent random variables X_k with distributions $P(X_k = 1) = p_k = 1 - P(X_k = 0)$. If the probabilities p_k depend on n in such a way that the largest p_k tends to zero but the sum $p_1 + \dots + p_n = \lambda$ remains constant, then in the limit as $n \rightarrow \infty$, S_n has a Poisson distribution with parameter λ (Feller, 1968, p. 282). Thus for large n and moderate values of λ , the distribution of S_n can be approximated by a Poisson distribution. This justification of a Poisson model has appeal for modeling failures of such parts as landing gear, for which it is more natural to assume that a failure will occur with some fixed probability on each mission than to assume a Poisson failure distribution for landing gear directly.

Both of these substantive considerations rely on an assumption of independence. In some situations this may be clearly inappropriate, for example, for parts that age rapidly or have very high failure rates

(Crawford, 1981, p. 11). However, loosening the independence assumption would require extensive reworking of the probability theory underlying the failure models, and currently the independence assumption is not believed to be inaccurate enough to justify this reworking. Henceforth, we will assume that failures follow a compound Poisson distribution.

At this point, several modeling issues can be raised:

1. As already noted, independence of numbers of failures in disjoint time intervals is assumed. This will not be pursued further.
2. Is the failure distribution really stationary in peacetime? For example, is there seasonal variation in the failure behavior of certain kinds of parts? Available data indicate that there is some seasonal variation, and it is not difficult to imagine that there would be some in extreme climates. Stationarity will henceforth be assumed.
3. Even assuming independence, is there any reason other than convenience to assume that the appropriate compound Poisson is a negative binomial and not some other compound Poisson? There has been some investigation indicating that, for real aircraft failure data, the variance-to-mean ratio increases with the mean, which is not a property of the negative binomial although it is a property of other compound Poisson distributions. The actual import of this apparent finding is unclear, because it may simply be an artifact of the estimator used.
4. The relationship between mean demands and flying hours is not well understood. First, it is not at all clear that flying hours are an appropriate "clock"--e.g., for landing gear what matters is not how long the plane is in the air, but how often it lands; also, some radar parts spend substantial amounts of time switched on and running while the plane is on the ground, so that flying hours understate the actual intensity of use. Second, there is no particular reason to assume that the relationship between flying hours and mean demands goes through the origin. If the planes simply sat in the hangars, some failures would occur anyway. Also, if this relationship were nonlinear with a positive second derivative, and it was desired to approximate the relationship for higher numbers of flying hours, a linear approximation might be appropriate, but its intercept would be negative. Third, there is no particular reason to assume that mean demands are a linear nondecreasing function of flying hours. On the contrary, there is evidence that high sortie rates can actually improve the "health" of some aircraft parts. A study by Crawford and Kamins (forthcoming) of objective measures of the health of components in the fire control and weapons delivery system of F-16s, gathered during a surge of aircraft activity during an exercise, found that reported rates of malfunctions, as

measured by the aircraft systems, actually declined compared with rates reported in periods of lower activity before and after the surge.

5. The available data are somewhat less than ideal. The Air Force Logistics Command accumulates failure data for each part over all bases worldwide, and disaggregated data are not currently available, as far as I know. The base data available for this study were collected at Rand for 20 parts with fairly high demand rates, on five airbases, over at most 13 quarters. Among the problems with these data were: (a) "managed demand," or possible changes in reported numbers of failures associated with management decisions related to reporting or servicing; and (b) a small number of observations for each part/base combination.
6. There was a manifest lack of enthusiasm among data collectors (mechanics), the utility of the data not having been strongly impressed on them.

This Note will consider only the third of these issues, the nature of the compounding distribution, and the fourth, the relationship between flying hours and the mean numbers of failures. This emphasis does not imply a judgment that the other issues are unimportant. In fact, they are so poorly understood that their importance cannot be assessed satisfactorily. But in the consideration of competing spare parts supply systems, it is essential to understand how well we can predict levels of parts failures under wartime conditions; issues that are not well understood cannot be dismissed.

II. SOME NOTES ON AN ESTIMATOR OF VARIANCE-TO-MEAN RATIO

INTRODUCTION

This section describes some properties of an estimator of variance-to-mean ratio called V. Let

X_i = number of demands in period i ($i = 1, 2, \dots, n$),
 $N = \sum X_i$ = total number of demands over all n periods,
 f_i = number of flying hours in period i ($i = 1, 2, \dots, n$),
 $T = \sum f_i$ = total number of flying hours in all periods.

If $N > 0$, V is defined to be

$$V = S^2 / \hat{\lambda},$$

where $\hat{\lambda} = N/T$ and

$$(n - 1)S^2 = \sum (X_i - f_i \hat{\lambda})^2 / f_i = \sum f_i (X_i / f_i - \hat{\lambda})^2.$$

If $N = 0$, both numerator and denominator of V are zero, and V is defined to have the value 1 for reasons that will become clear below.

The estimator V is $1/(n - 1)$ times the so-called "index of dispersion" defined by

$$D = \sum (X_i - f_i \hat{\lambda})^2 / f_i \hat{\lambda}.$$

Note that D results from replacing $E(X_i)$ by $f_i \hat{\lambda}$ in the expression

$$\chi^2 = \sum (X_i - E(X_i))^2 / E(X_i),$$

and that D is also the χ^2 statistic for testing the hypothesis that the observations X_i have Poisson distributions. In this case, the conditional distribution of (X_1, X_2, \dots, X_n) given N is multinomial

with parameters $p_i = f_i/T$. Hence, under the Poisson hypothesis the well-known asymptotic result gives the conditional distribution of D given N as approximately χ^2 with $n - 1$ degrees of freedom.

This study was prompted by an examination of the data mentioned in Sec. I in which V was used. Judged against prevailing beliefs about values of ρ for actual part failure distributions, the results seemed quite unusual--43 of the hundred values of V calculated were greater than 5, 15 exceeded 15, and one was greater than 95. This appeared to contradict what was known about the properties of V for values of ρ generally considered reasonable and suggested that these large values of V were indicating model failures as well as larger than expected variability. To see whether model failures could result in values of V similar to those observed, and to study further properties of V under the more favorable assumptions described in Sec. I, V was examined under several sets of assumptions, three sets of which will be discussed here: that the number of failures in a period of given length has (1) a Poisson distribution with parameter λf , (2) a negative binomial distribution with parameters $r = \lambda f/(1 - \rho)$ and $p = 1/\rho$, and (3) a Poisson distribution with parameter $\alpha + \beta f$, $\alpha \neq 0$.

Apart from their relevance to the part failure problem, the results of (1) are relevant to inference for nonhomogeneous Poisson processes in other situations. In (3), it is shown that simple model failure alone could not reasonably have produced the unusual observed values of V , although plots of failures against flying hours suggest that including an intercept term in the negative binomial mean is desirable in some cases. The main result in (2) is that V is biased low, with the bias increasing as ρ gets larger relative to λT . This implies that even if the negative binomial model with mean λf adequately characterizes the process generating part failures, the true variability is underestimated on the average by V , with the largest underestimation occurring for the largest values of ρ .

PROPERTIES OF V UNDER POISSON ASSUMPTIONS

The properties of V are examined here under Poisson assumptions. Expected values and variances are derived for various values of n and patterns of flying hours, and Monte Carlo simulations of the distribution of V are displayed and discussed.

The assumptions (henceforth referred to as "the Poisson assumptions") are as follows: if X_i is the number of demands for a given part in the ith period, and f_i the flying hours for the ith period, then assume

- (1) X_i is a random variable with a Poisson distribution, having mean λf_i . That is, demands are stationary over periods, with a constant expected rate of demands per flying hour.
- (2) The counts for different periods are independent.

Whether the Poisson assumptions hold or not, the denominator of V is unbiased for λ . Under the Poisson assumptions, the numerator is also unbiased for λ , and V has expectation 1. Indeed, $E(V|N) = 1$ for all values of $N > 0$. These results stem from the fact that, given N, the conditional distribution of X_i is binomial with parameters N and $p_i = f_i/T$. It follows that $E(X_i|N) = Np_i = f_i\hat{\lambda}$ and

$$E[(X_i - f_i\hat{\lambda})^2|N] = \text{var}(X_i|N) = Np_i(1 - p_i) = f_i\hat{\lambda}(1 - p_i).$$

Hence, the conditional expectation of the numerator of V is

$$E(S^2|N) = (n - 1)^{-1} \sum \hat{\lambda}(1 - f_i/T) = \hat{\lambda}$$

and $E(V|N) = E(S^2/\hat{\lambda}|N) = 1$. Taking expectations yields $E(S^2) = \lambda$ and $E(V) = 1$.

This generalizes a result by Rao and Chakravarti (1956, pp. 265-266), for the case of equal f_i . Note that this result does not require that the f_i be known. Appendix A contains a derivation of $E(V)$ under an assumption that $E(X_i) = \lambda_i$, which may be of use in other sensitivity analyses.

The above results do not hold in general; i.e., V is not usually unbiased for the variance-to-mean ratio. In particular, it is biased in the negative binomial case, as will be seen later.

Both the conditional and unconditional variances of V are of interest for assessing the reliability of V as an estimator of ρ . As is shown below, the conditional variance of V , given a nonzero value of N , is

$$\text{var}(V|N) = 2(n-1)^{-1}(1 - C/N),$$

where $C = 1 + [n^2 - T \sum (1/f_i)]/2(n-1)$. If $f_1 = f_2 = \dots = f_n$, the constant C is equal to one, and this is the largest possible value of C . This follows from noting that $\sum (1/f_i) \geq \sum (1/\bar{f}) = n^2/T$ by Jensen's Inequality, so that

$$\text{var}(V|N) \geq 2(n-1)^{-1}(1 - 1/N),$$

with equality holding if and only if the f_i are equal. This lower bound should serve as a useful approximation in applications where the f_i are approximately equal.

The unconditional variance of V is obtained by using the fact that

$$\text{var}(V) = \text{var}[E(V|N)] + E[\text{var}(V|N)],$$

and noting that $\text{var}(V|N) = 0$ when $N = 0$ gives

$$\begin{aligned} \text{var}(V) &= 2(n-1)^{-1} [1 - CE(N^{-1}|N > 0)]P(N > 0) \\ &= 2(n-1)^{-1} [1 - e^{-\theta} - CH(\theta)] \end{aligned}$$

where $\theta = T\lambda$ and

$$H(\theta) = \sum_{k=1}^{\infty} \theta^k e^{-\theta} / k \cdot k! = e^{-\theta} \int_0^{\theta} (e^u - 1)/u \, du.$$

Since $E(N^{-1}|N > 0) > 1/E(N|N > 0) = (1 - e^{-\theta})/\theta$ by Jensen's Inequality,

$$H(\theta) > (1 - e^{-\theta})^2/\theta.$$

This provides an upper bound for $\text{var}(V)$ that also serves as a good approximation for large values of θ . For a table of values of $H(\theta)$ for θ between 0.01 and 20, see Grab and Savage (1954). The approximation $H(\theta) \approx 1/(\theta-1)$ serves quite well for $\theta > 5$, and this approximation was used in the work reported below.

To derive $\text{var}(V|N)$ for $N > 0$, it is convenient to set

$$Q = \sum (X_i - f_i \hat{\lambda})^2 / f_i,$$

so that $V = Q/(n - 1)\hat{\lambda}$, and

$$\text{var}(V|N) = \text{var}(Q|N)/(n - 1)^2 \hat{\lambda}^2.$$

Hence, it suffices to show that, for $N > 0$,

$$\text{var}(Q|N) = 2(n - 1)(1 - C/N)\hat{\lambda}^2$$

where C is as above. By using the representation

$$Q = \sum (X_i - f_i \lambda)^2 / f_i - T(\hat{\lambda} - \lambda)^2,$$

which implies that

$$[Q + T(\hat{\lambda} - \lambda)^2]^2 = \sum_{i \neq j} U_i^2 + \sum U_i U_j$$

where $U_i = (X_i - f_i \lambda)^2 / f_i$, it is straightforward to show that

$$E(Q^2) = \lambda \{ \sum (1/f_i) + (1 - 2n)T \} + \lambda^2 (n^2 - 1).$$

Then

$$E(Q^2|N) = \hat{\lambda}[\Sigma(1/f_i) + (1 - 2n)/T] + (\hat{\lambda}^2 - \hat{\lambda}/T)(n^2 - 1),$$

since $\hat{\lambda}$ is unbiased for λ , $\hat{\lambda}^2 - \hat{\lambda}/T$ is unbiased for λ^2 , and N is a complete statistic, which implies that the unbiased estimator of $E(Q^2)$ that depends on N is unique (Lindgren, 1976, p. 256). To complete the derivation, one can use the formula

$$\text{var}(Q|N) = E(Q^2|N) - [E(Q|N)]^2$$

and the fact that $E(Q|N) = (n - 1)\hat{\lambda}$.

To show how the values of $\text{var}(V)$ depend on λ , n , and $\{f_i\}$, four patterns of the values $\{f_j\}$ were chosen so that the average was always 2500, and thus their sum was always $2500n$. These four patterns are as follows:

- | | | |
|-----|--|--|
| (1) | $f_j = 2500$ | $j = 1, \dots, n$ |
| (2) | $f_j = 2500$
$f_1 = 1000$
$f_n = 4000$ | $j = 2, \dots, n - 1$ |
| (3) | $f_j = 1950 + 100j$
$f_j = 1450 + 100j$
$f_j = 1225 + 50j$ | for $n = 10$
for $n = 20$
for $n = 50$ |
| (4) | $f_j = 25 + 450j$
$f_j = -20 + 240j$
$f_j = -50 + 100j$ | for $n = 10$
for $n = 20$
for $n = 50$ |

The entries in the table below are $\text{var}(V)$.

As this table shows, the effect of heterogeneity in flying hours on $\text{var}(V)$ is more pronounced for smaller values of λ . Note that pattern (1) represents the case $C = 1$, and that the entries for that case serve as a lower bound and approximation for the other cases.

f-pattern	$\lambda = 0.01$			$\lambda = 0.001$		
	n = 10	20	50	n = 10	20	50
1	0.221	0.105	0.041	0.213	0.103	0.040
2	0.229	0.105	0.041	0.290	0.103	0.040
3	0.222	0.106	0.041	0.224	0.115	0.047
4	0.242	0.115	0.048	0.432	0.203	0.116

The histograms of the observed values of V from selected Monte Carlo runs appear in Fig. 2. The results for $f_j = 2500$ and $\lambda = 0.001$ were essentially the same as those for $f_j = 2500$ and $\lambda = 0.01$, so they were not included. One thing to note here is the difference between the last two histograms. Decreasing λ from 0.01 to 0.001 flattens the mode and moves some of the probability to each tail.

NEGATIVE BINOMIAL

The negative binomial may be characterized by its mean and the ratio of its variance to its mean. This characterization is of interest because the negative binomial distribution tends to the Poisson as the variance-to-mean ratio tends to 1, and thus in a rough sense the variance-to-mean ratio describes the magnitude of the departure from the Poisson assumption.

In particular, it is assumed that X_j has a negative binomial distribution with mean λf_j and variance-to-mean ratio p , so that

$$P(X_j = k) = \binom{r_j + k - 1}{k} (1 - p)^k p^{r_j} \quad k = 0, 1, 2, \dots,$$

where $r_j = \lambda f_j / (p - 1)$, $p = 1/\rho$.

The mean of V may be derived by noting that $E(Q) = (n - 1)\lambda/p$, for $Q = (n - 1)S^2$ and S^2 defined as in the introduction to Sec. II. Thus, $E(S^2) = \lambda/p$. Because $\hat{\lambda}(R + N)/(R + 1)$, where $R = \sum r_i$, also has expectation λ/p , it follows that this expression is equal to $E(S^2|N)$ by the completeness of N . Hence, for $N > 0$,

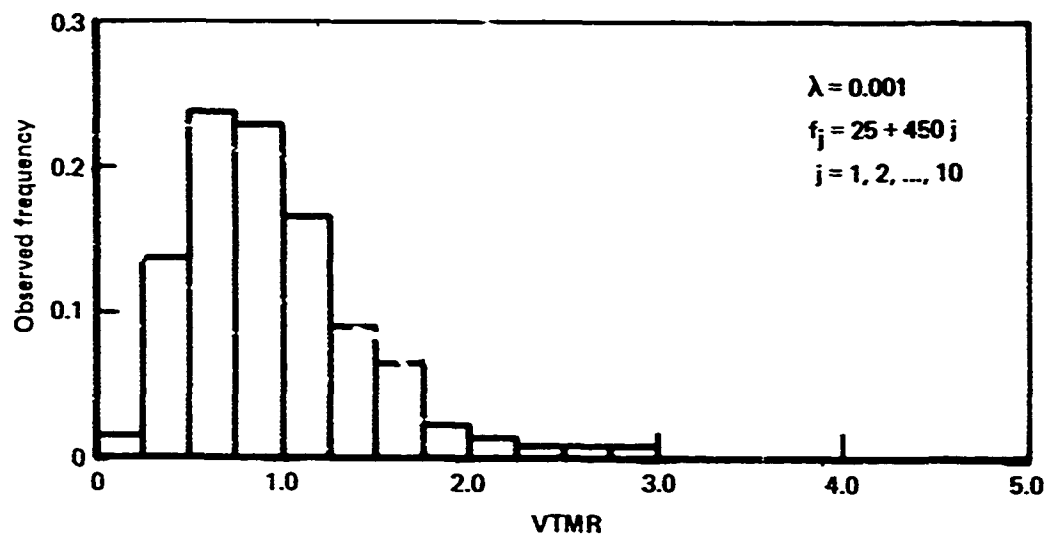
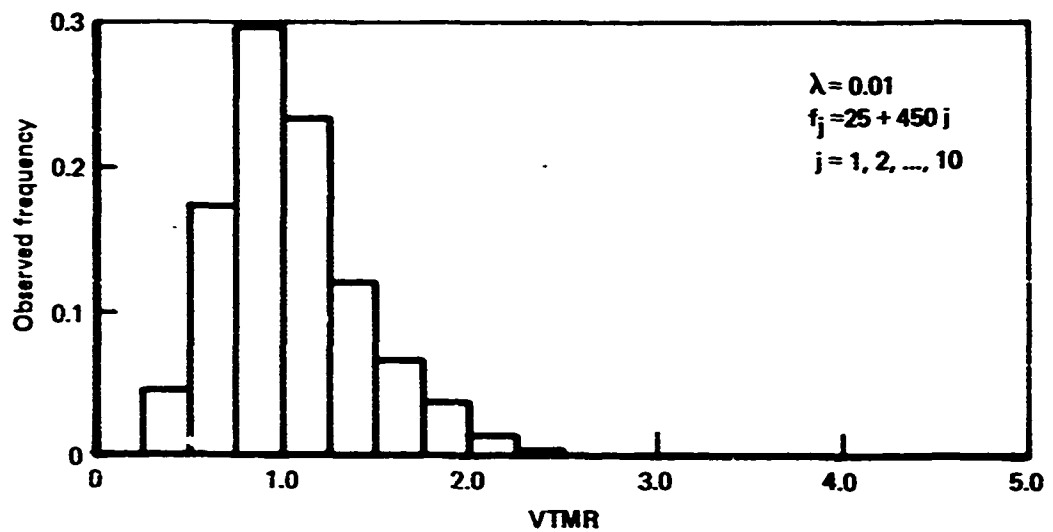
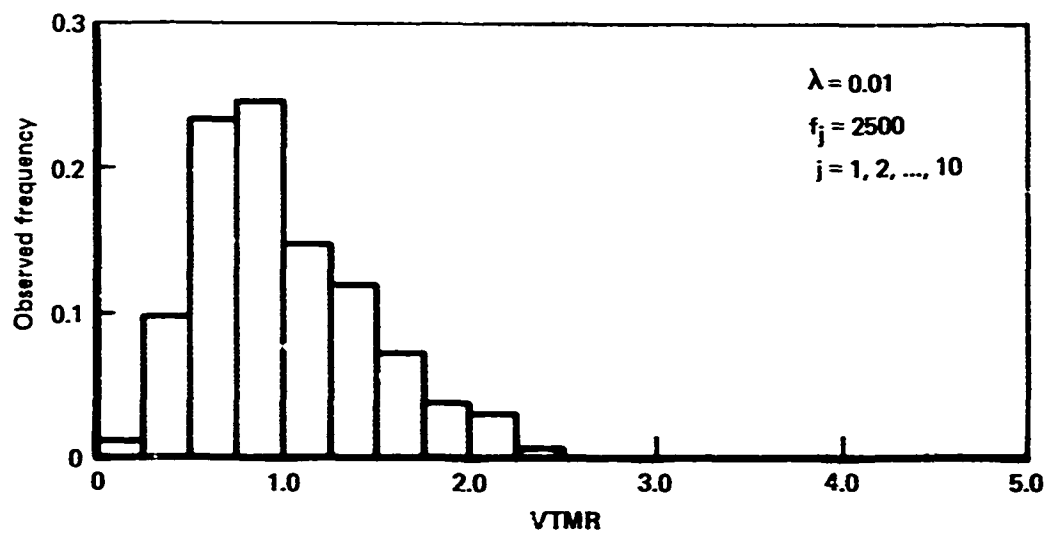


Fig. 2 - Histograms of V under Poisson assumptions

$$E(V|N) = E(S^2/\hat{\lambda}|N) = (R + N)/(R + 1),$$

and, as before, $V = 1$ for $N = 0$ by definition. This can also be written as

$$E(V|N) = 1 + (\rho - 1)(N - 1)/(\lambda T + \rho - 1).$$

From either of these expressions, it follows immediately that

$$E(V) = \rho(R + \rho^{-R-1})/(R + 1),$$

which is less than ρ for $\rho > 1$.

This bias in V accounts for at least part of the apparent dependence of variance-to-mean ratio on the mean. As the mean increases--i.e., as λT increases--the bias will decrease, and $E(V)$ will increase. Thus, even if all of the assumptions of the negative binomial model were satisfied, if V was used to estimate the variance-to-mean ratio, the estimates would indicate that V increases with the mean.

Although $\text{var}(V)$ and $\text{var}(V|N)$ are tractable under the assumptions of this subsection using straightforward algebra, the formulas are very complicated and offer little insight.

If μ and T are assumed to be large, the distribution of V can be approximated by the same expression when $\hat{\lambda}$ is replaced by λ , which will be called V^* . It can be shown that in the Poisson case, with parameter λf ,

$$\text{var}(V^*) = [2n + \sum (1/f_i \lambda)]/(n - 1)^2,$$

whereas, in the negative binomial case,

$$\text{var}(V^*) = [2n/p^2 + (6q + p^2)\sum(1/f_i)/\lambda p^3]/(n - 1)^2,$$

where $q = 1 - p$. Thus, when the f_i are large, $\text{var}(V)$ in the negative binomial case is approximately $p^2 = 1/p^2$ times as large as the variance in the Poisson case.

This approximation should be used with caution. When computed for the values of n , λ , and $\{f_j\}$ used in the table of Poisson model variances and $p = 2, 5$, and 15 , the approximation was always greater than the exact value, by factors as large as 8 , with the error growing larger as λ decreases, p increases, or $\{f_j\}$ becomes more heterogeneous.

Some Monte Carlo samples were drawn from V 's distribution for various values of λ and p , two patterns of $\{f_j\}$, and $n = 10$. Selected histograms appear in Fig. 3(a-j). These figures are all in the same scale to facilitate making comparisons.

Some generalizations can be made from the histograms about the effect on V 's distribution of λ , p , and $\{f_j\}$. First, decreasing λ from 0.01 to 0.001 shifts some probability to smaller values of V , apparently away from middle values, because the upper tails do not seem to differ. Second, changing from constant to nonconstant f_j also seems to shift probability away from middle values, but this is divided between small values and the upper tail. That is, changing away from equal flying hours makes more probable both small values and very large values.

In terms of the above trends, the Poisson case can simply be thought of as the case with $p = 1$.

POISSON MODEL, WITH $E(X_i) = \alpha + \beta f_i$

There are two reasons why it might be desirable to allow the mean of the failure distribution to be a more general function of flying hours than λf . It is possible that some parts would fail on planes that did not fly at all,¹ which implies that the function relating mean failures to flying hours can be approximated by lines with nonzero intercepts, over suitably restricted ranges of flying hours. If flying hours are suitably restricted, such approximations could have either positive or negative intercepts, and Fig. 4, drawn from our Air Force data, shows that both of these possibilities can occur, with Fig. 4a suggesting a positive intercept (and a negative slope), and Fig. 4b suggesting a negative intercept.

¹Disuse, especially in extreme climates, may contribute to a significant number of failures.

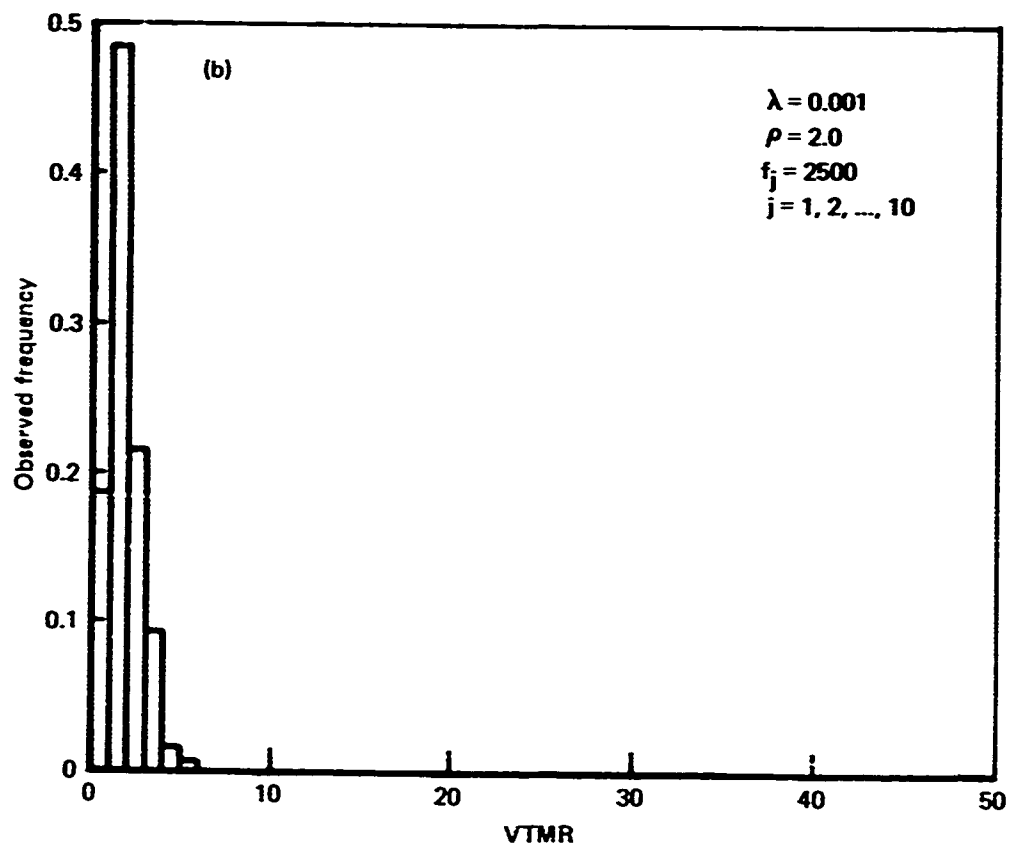
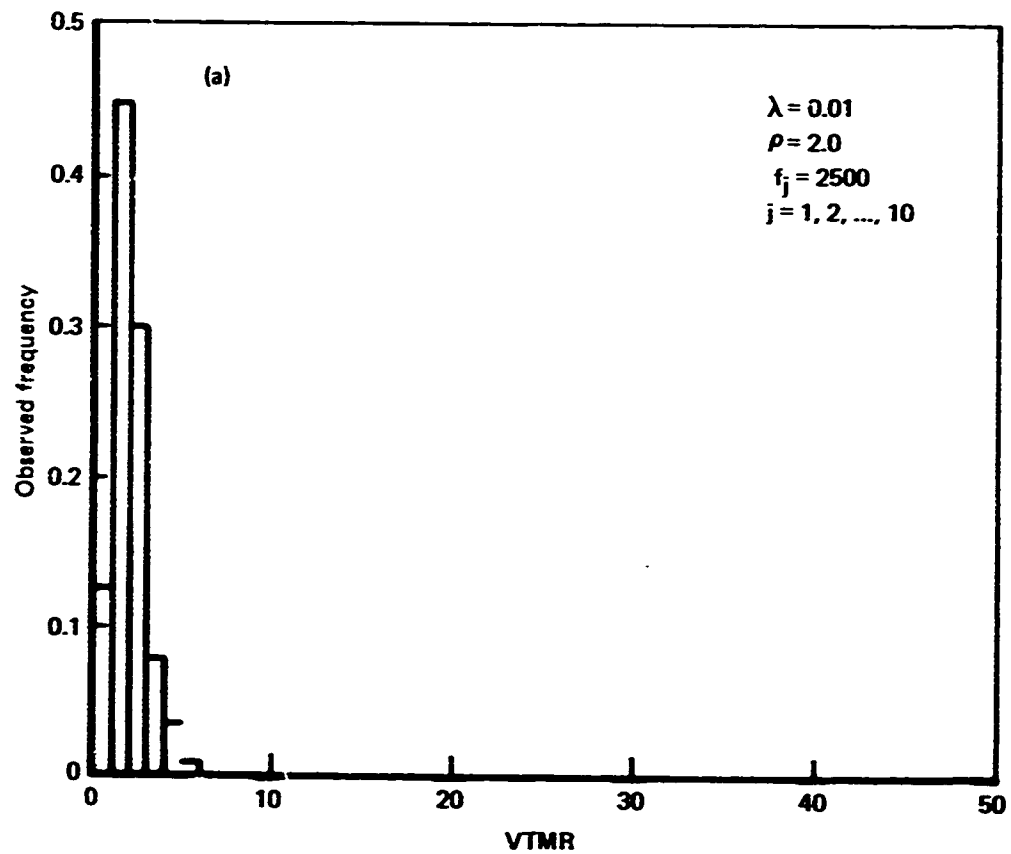


Fig. 3 — Histograms of V under negative binomial assumptions

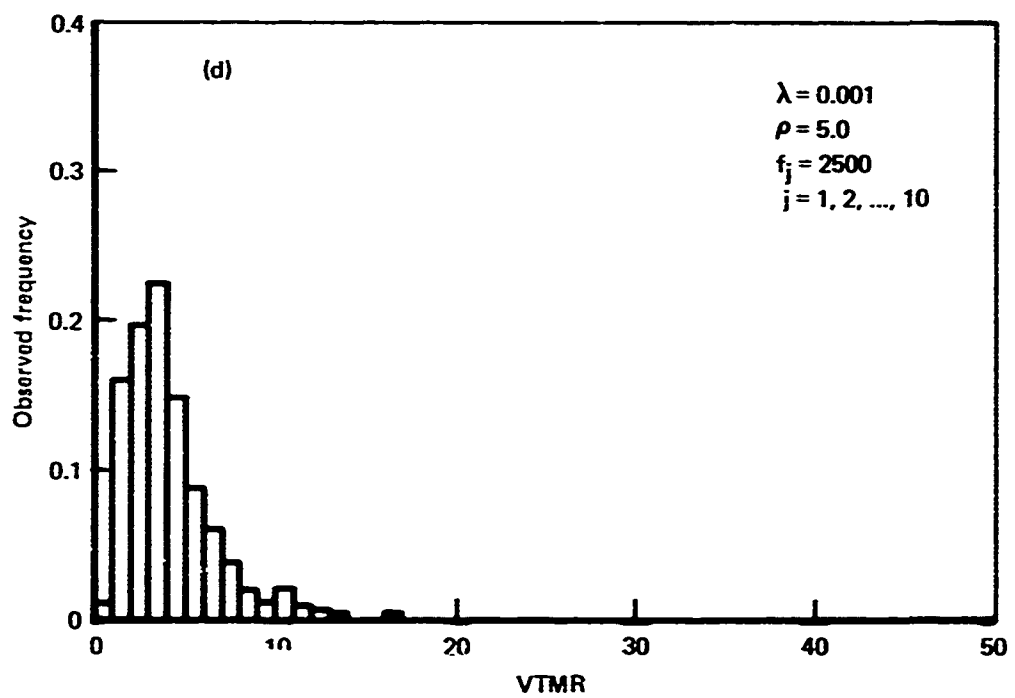
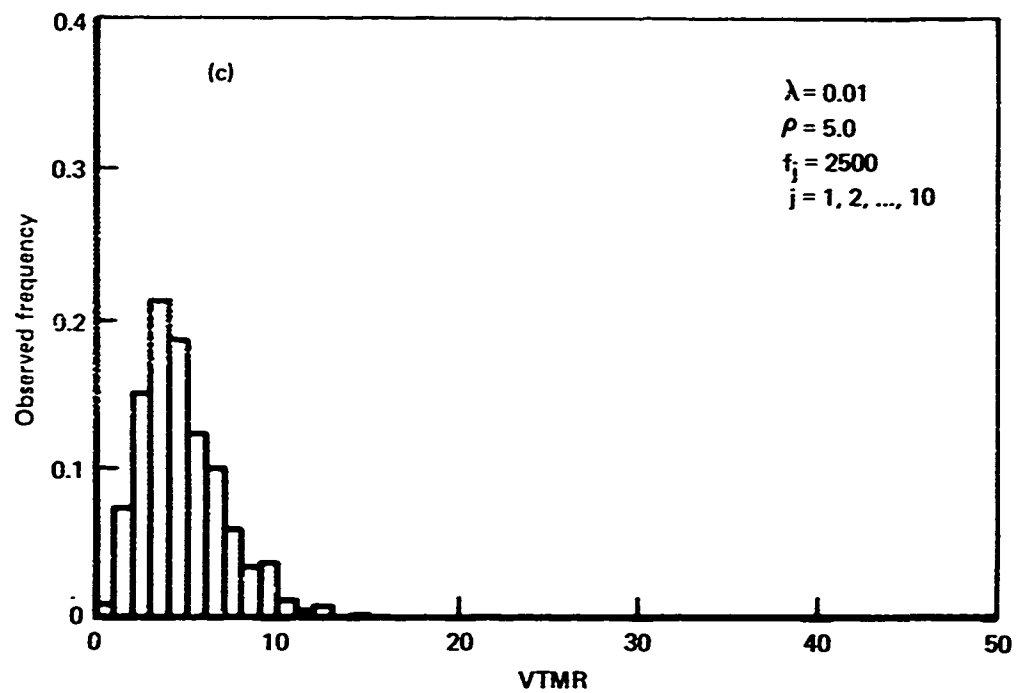


Fig. 3 - (continued)

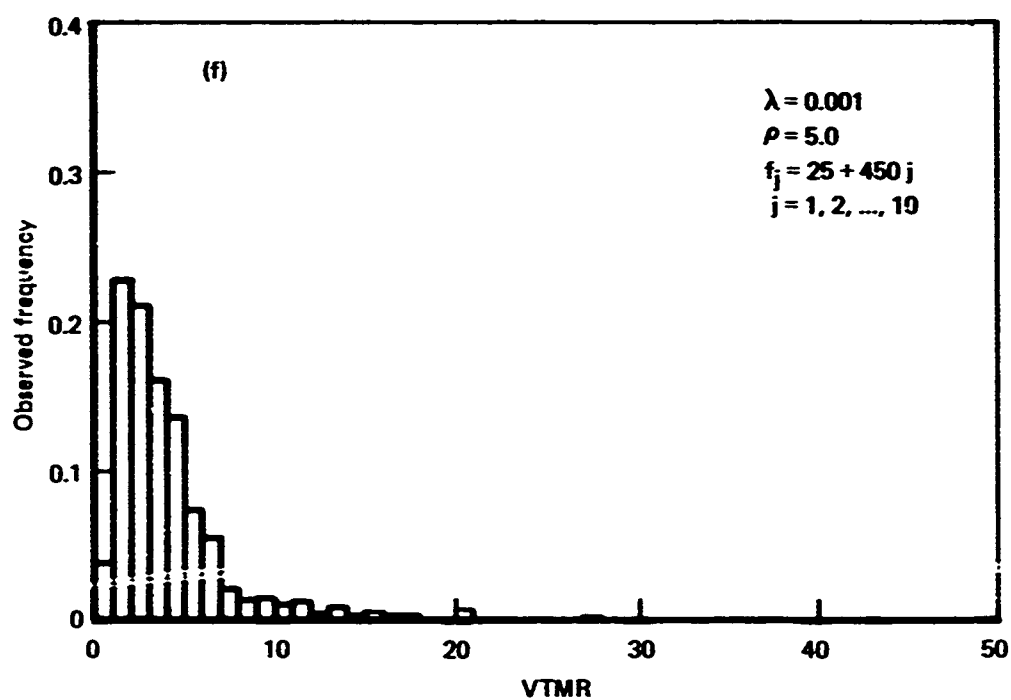
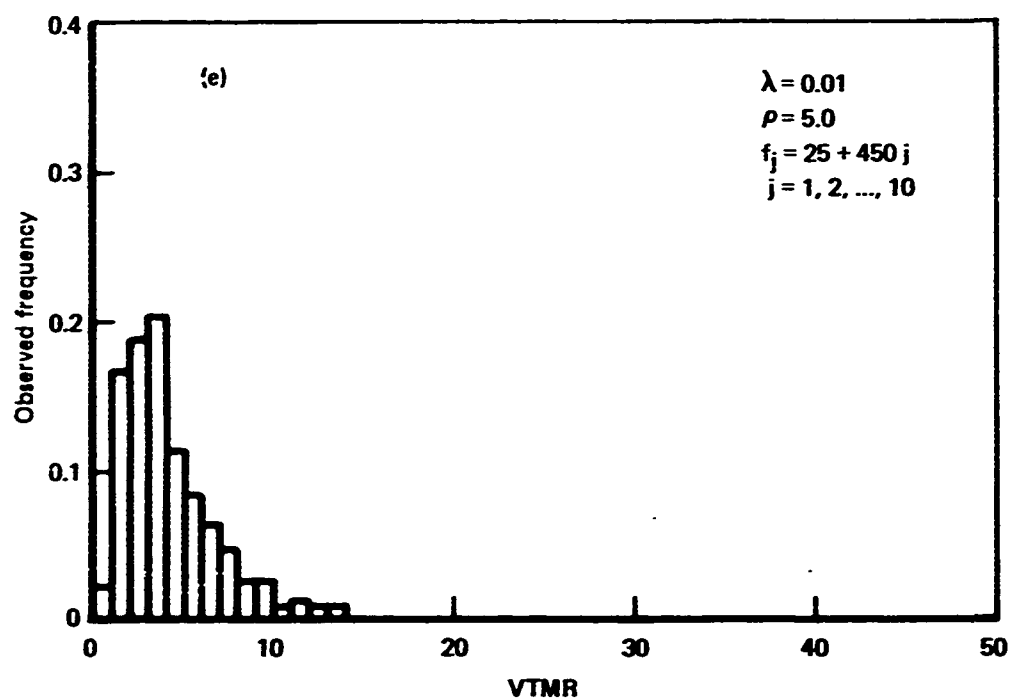


Fig. 3 - (continued)

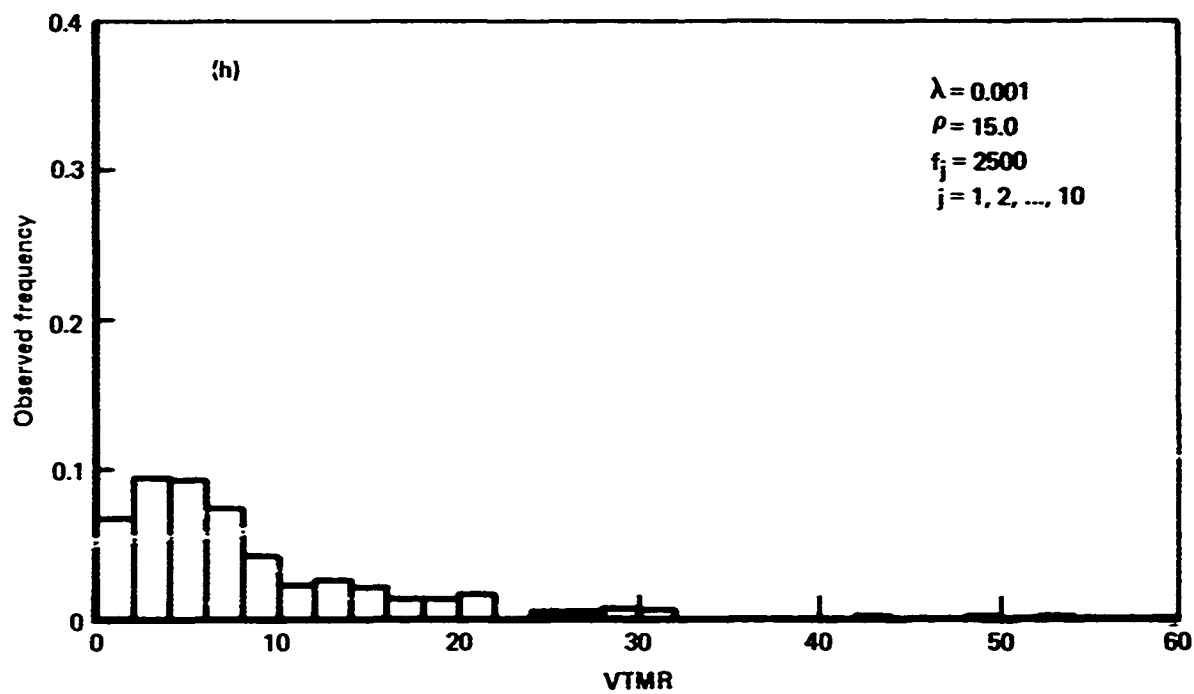
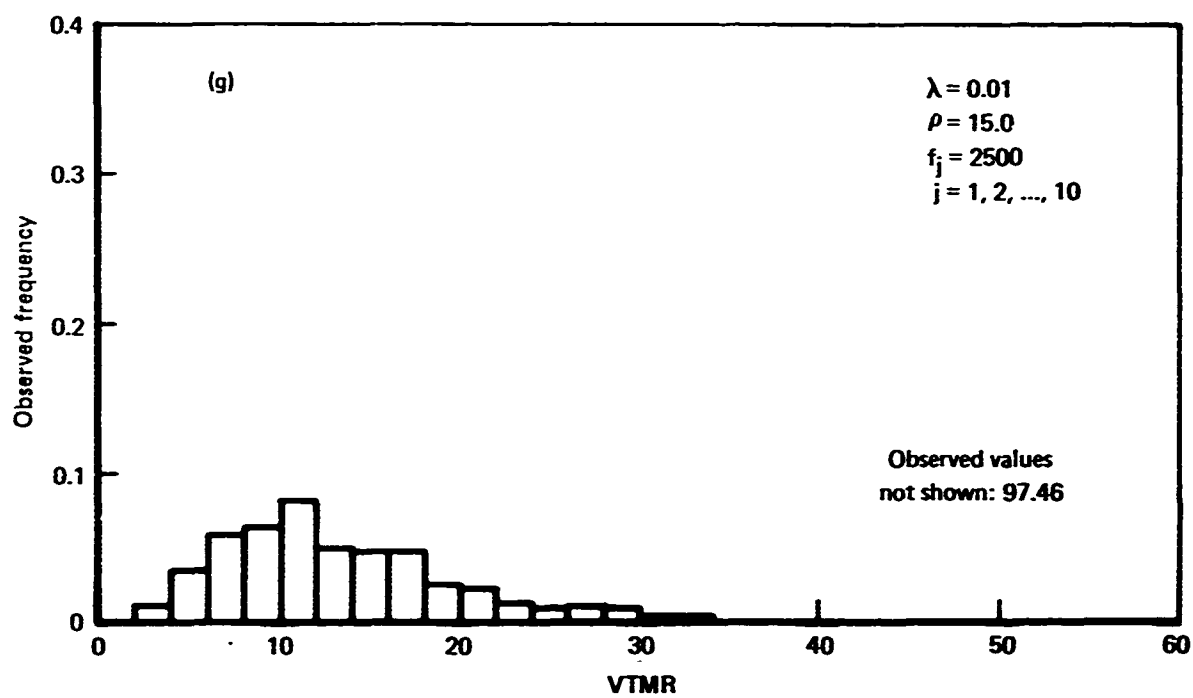


Fig. 3 - (continued)

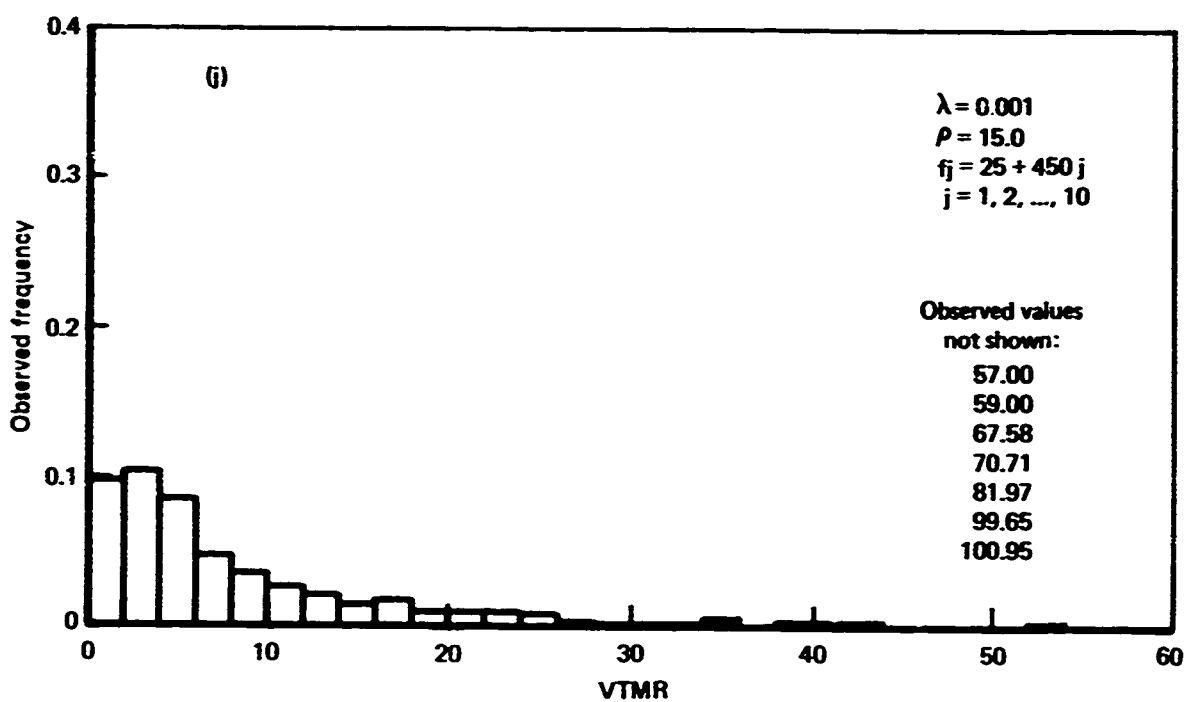
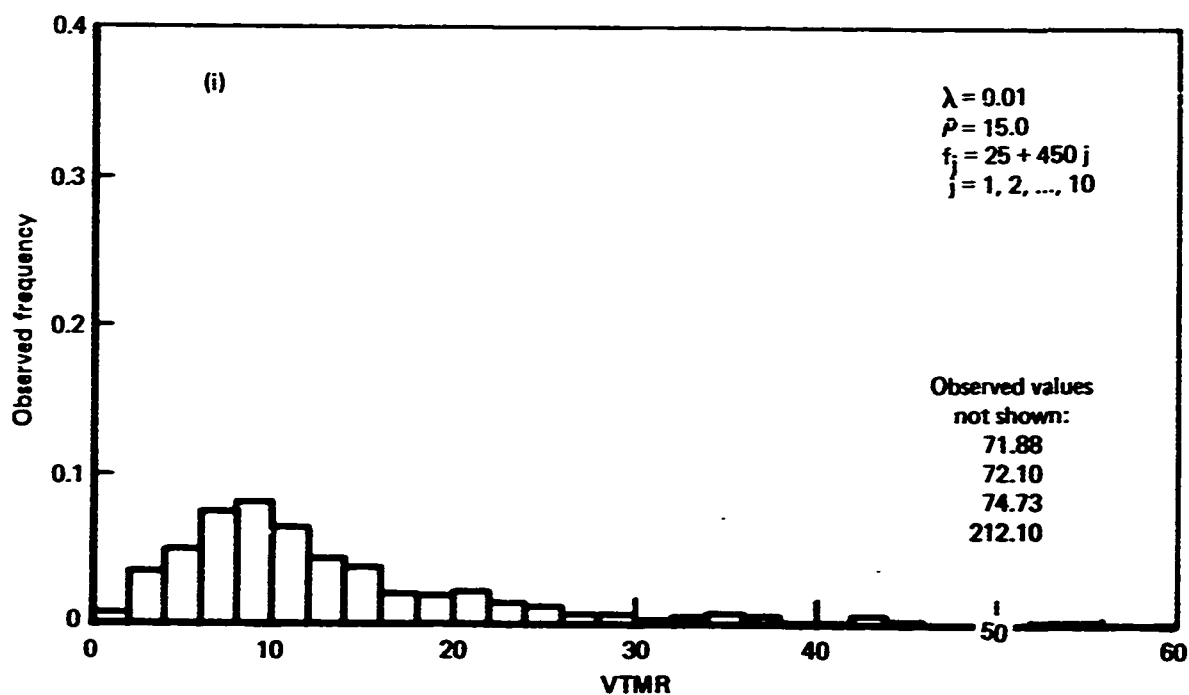


Fig. 3 - (continued)

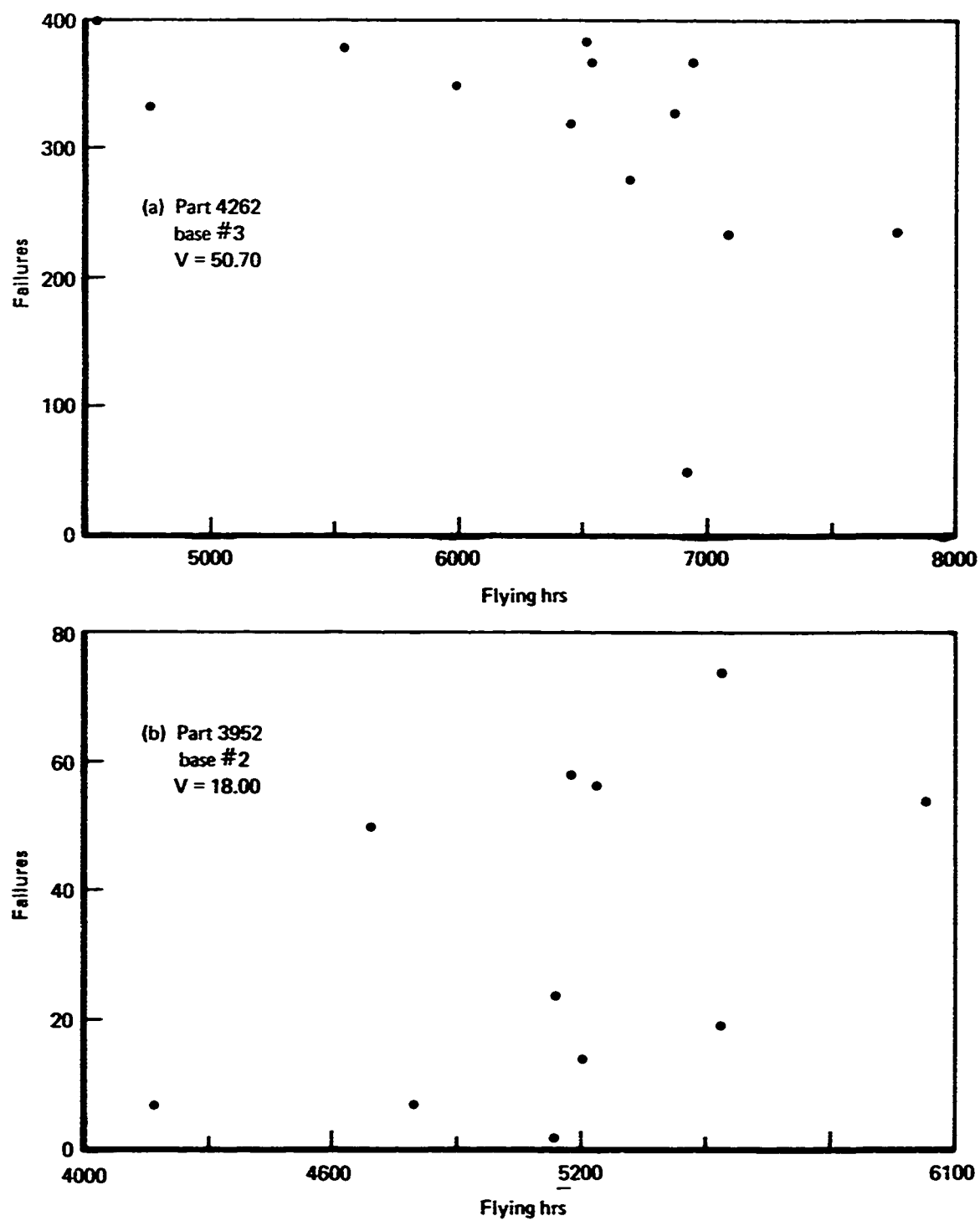


Fig. 4 — Flying hours versus failures for Air Force data

From these and other plots, it is reasonable to postulate a model of Poisson failure counts with mean equal to $\alpha + \beta f$ for α not necessarily zero. But could observations from such a model have produced V values like those observed, thus providing an explanation for their unexpected size? The results of this subsection indicate that they could not reasonably be expected to do so in all the cases examined, and that more is needed to explain the large observed V values.

For the remainder of this subsection, the number of part failures in a given time period is assumed to be a Poisson random variable with parameter $\alpha + \beta f$. Under this assumption, the formula in Appendix A can be used to compute $E(V)$ as a function of α , β , n , and $\{f_j\}$. This formula is quite complex; plots of $E(V)$ as a function of β , for $n = 10$, for several values of α and for two sets of $\{f_j\}$ appear as Fig. 5(a-d). Because it is necessary that $\alpha + \beta f_j > 0$, only certain values of β are permissible when $\alpha < 0$.

Appendix B contains a derivation of $\text{var}(V)$ as a function of α , β , n , and $\{f_j\}$. This formula is also quite complex, and plots of $\text{var}(V)$ as a function of β , for $n = 10$ and for the same values of α and $\{f_j\}$ as above, appear as Fig. 6(a-d).

A few trends are clear. $E(V)$ is larger for smaller values of β , more heterogeneous $\{f_j\}$, and larger $|\alpha|$, and the latter effect is much more pronounced for positive α . These trends also hold for $\text{var}(V)$, except that for negative α and more heterogeneous $\{f_j\}$, the size of β has little effect and that effect is not monotonic in β .

Poisson models with parameter $\alpha + \beta f_j$ were fitted to the 100 part-base combinations in our data set, using maximum likelihood estimation. The results of these fits are in Appendix C. They include the estimates $\hat{\alpha}$ and $\hat{\beta}$, their approximate standard errors and correlation obtained from the inverse of the information matrix, $E(V)$ and $\text{var}(V)$ computed from the aforementioned formulas using $\hat{\alpha}$ and $\hat{\beta}$, and the actual observed value of V . For seven part-base combinations, the fitted values α and β were such that $\hat{\alpha} + \hat{\beta} f_j < 0$ for some j ; $E(V)$ and $\text{var}(V)$ were not computed for those part-base combinations.

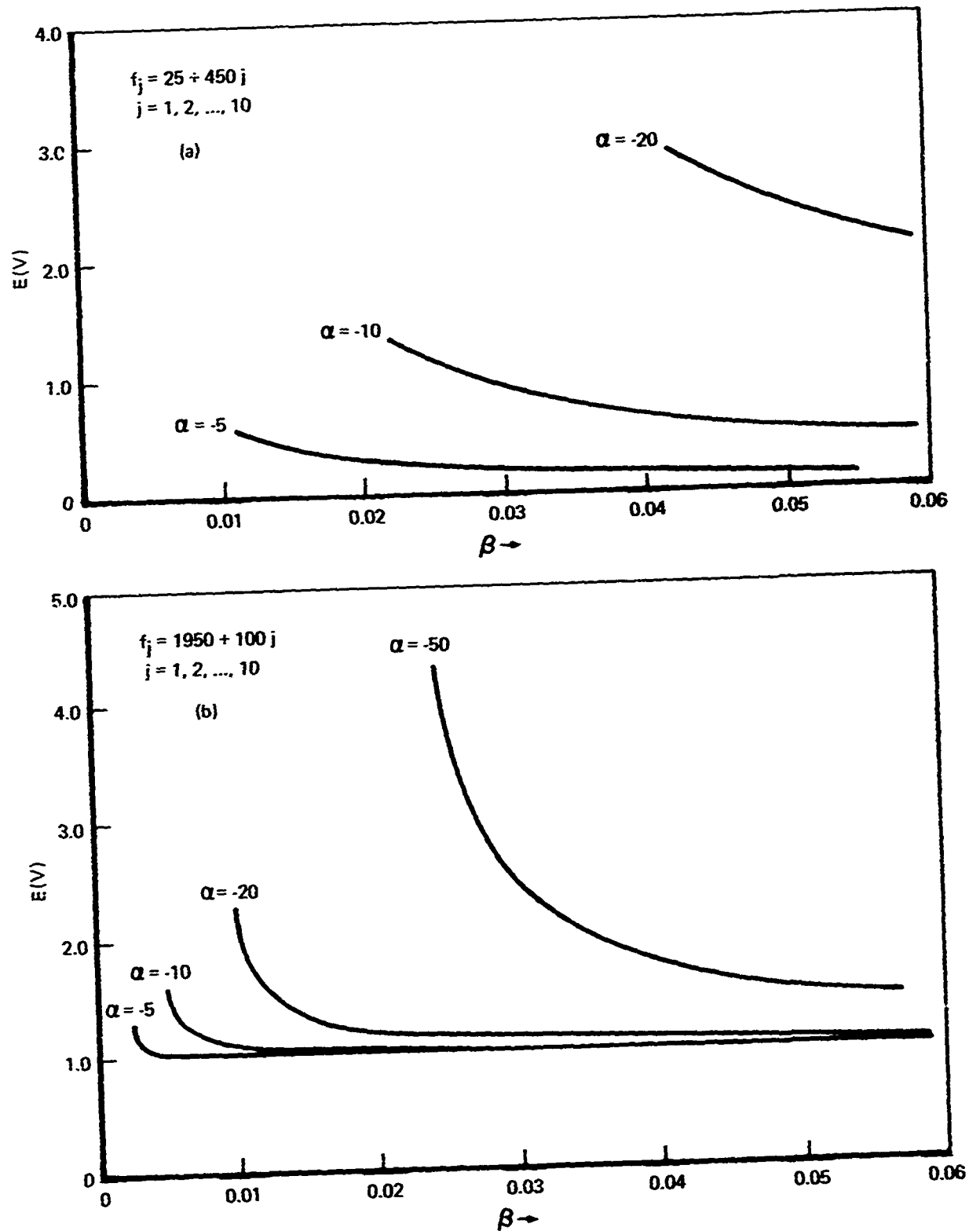


Fig. 5 — $E(V)$ as a function of α and β ; Poisson Model
with mean a linear function of flying hours

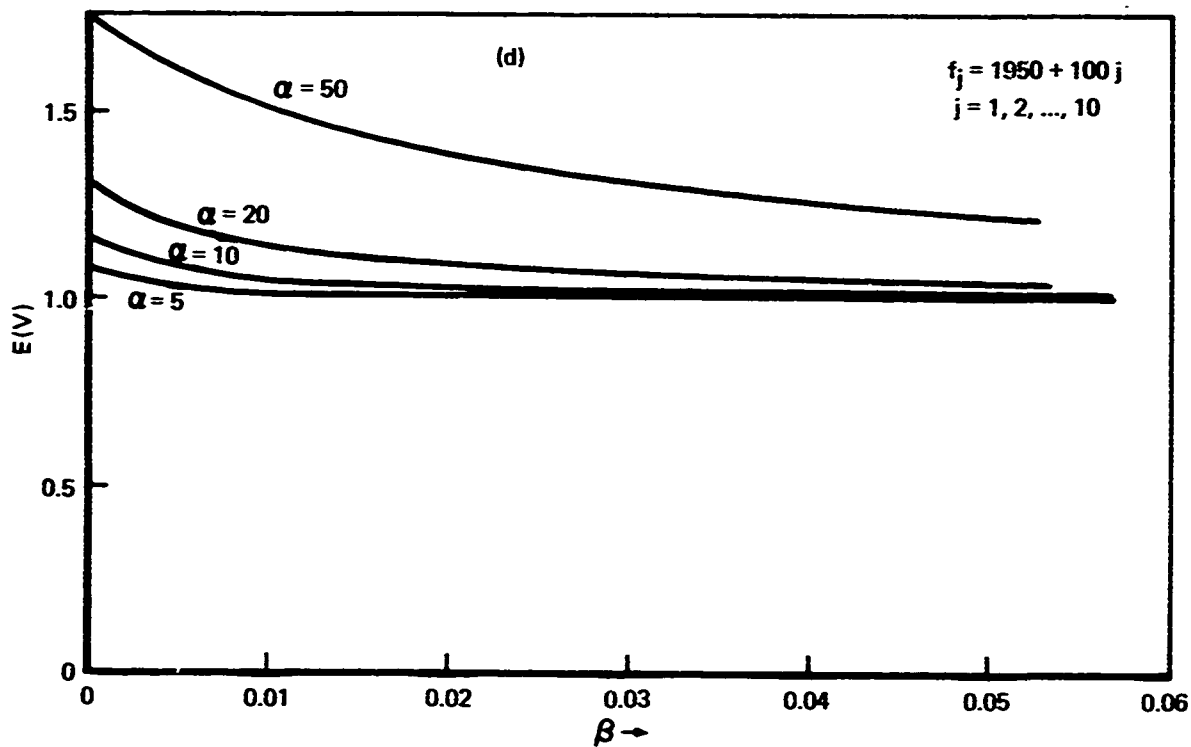
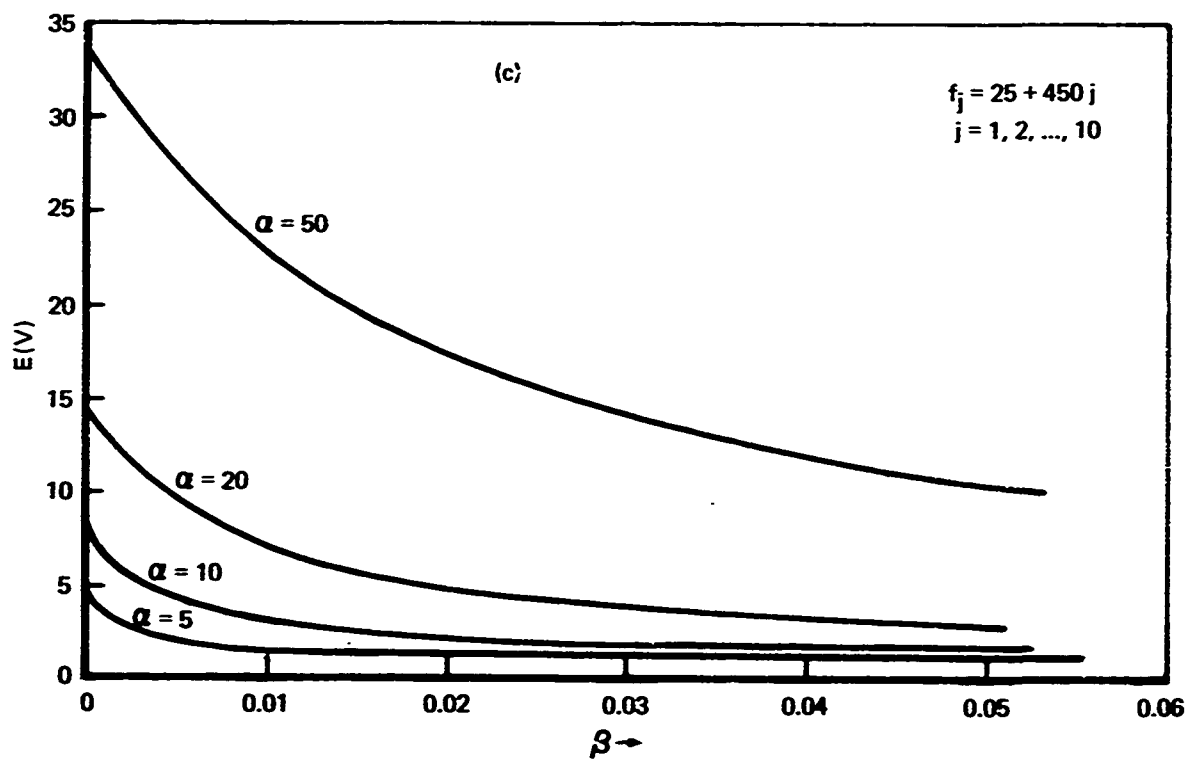


Fig. 5 - (continued)

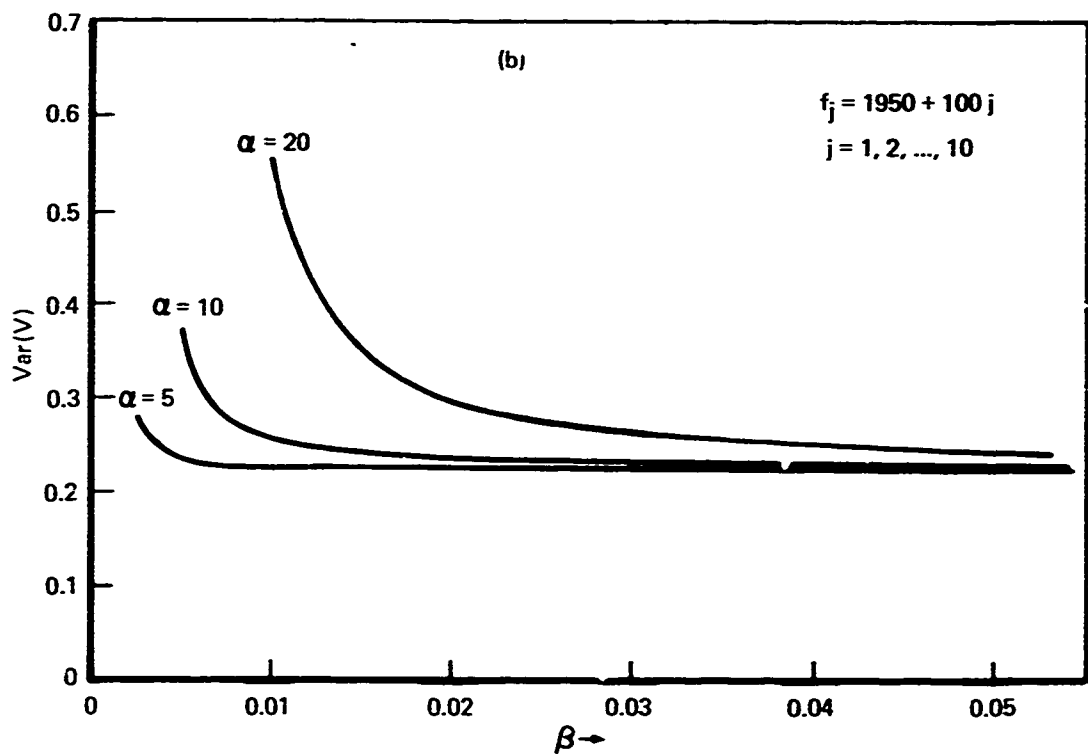
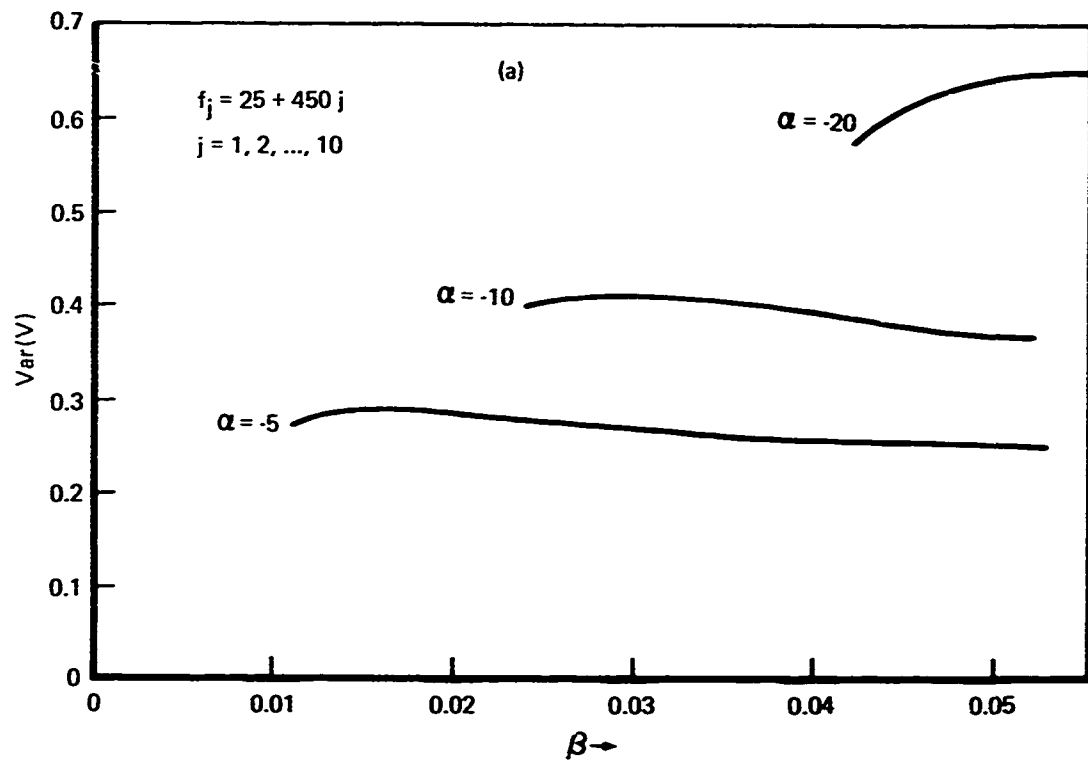


Fig. 6 - Var (V) as a function of α and β ; Poisson Model
with mean a linear function of flying hours

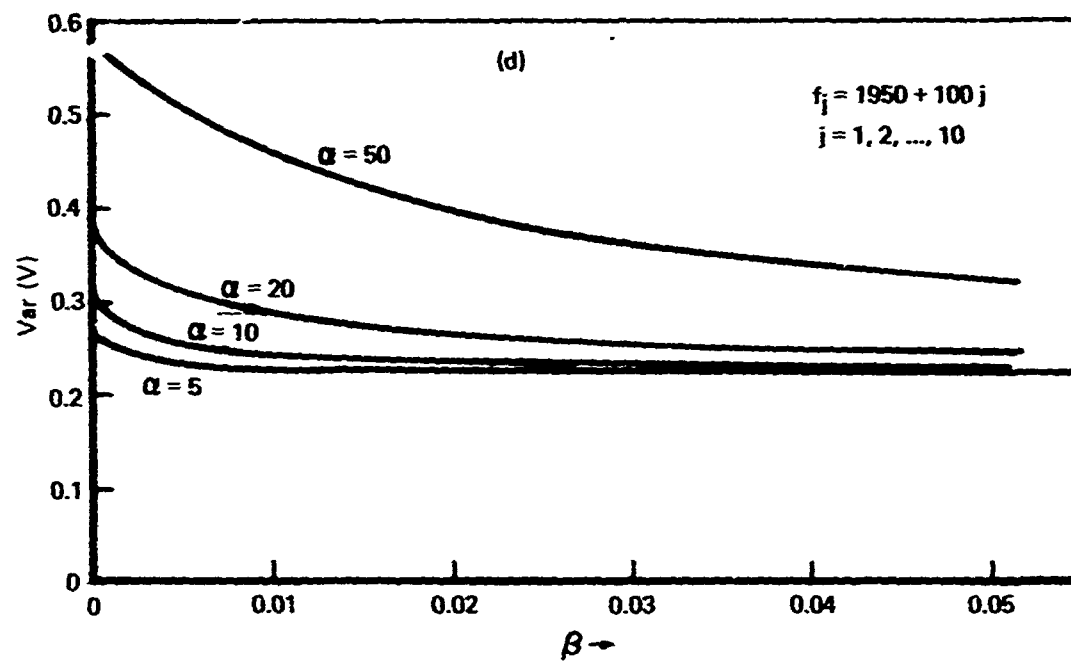
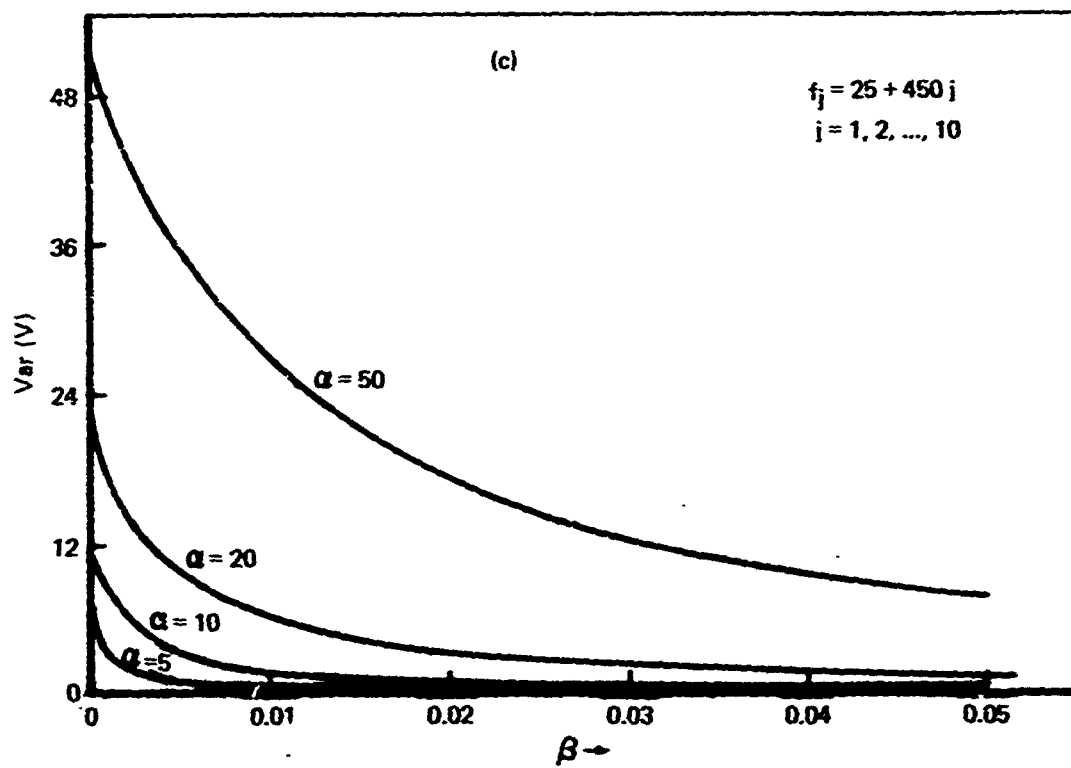


Fig. 6 - (continued)

A simulation was run for each of the 93 part-base combinations having all $\hat{\alpha} + \hat{\beta}f_j > 0$. Observations were generated for Poisson distributions with parameters $\hat{\alpha} + \hat{\beta}f_j$, $j = 1, 2, \dots, n$, and a pseudo-observation of V was computed from them and compared with the actual observed value of V 300 times for each admissible part-base combination. The proportion exceeding the actual observed value of V was recorded; these proportions appear as the right-most column in Appendix C.

The message of the simulation and of $E(V)$ and $\text{var}(V)$ is clear. Almost all of the large observed values of V are in the extreme upper tails of their simulated distributions, assuming that failures are Poisson with parameters $\alpha + \beta f_j$. Our data clearly indicate more variability than can be explained by simply allowing α to be nonzero.

However, the fitted values α and their approximate standard errors suggest that α should not be assumed to be zero. Although the preceding paragraphs cast doubt on the appropriateness of a Poisson model, and thus on the approximate standard errors derived from that model, it is striking nonetheless that of the 93 admissible part-base combinations, 20 have $|\hat{\alpha}| > 4 \text{ s.e.}(\hat{\alpha})$, 28 have $|\hat{\alpha}| > 3 \text{ s.e.}(\hat{\alpha})$, and 45 have $|\hat{\alpha}| > 2 \text{ s.e.}(\hat{\alpha})$. It is also suggestive that of the 41 admissible part-base combinations having an observed V in excess of 5, 17 have $|\hat{\alpha}| > 4 \text{ s.e.}(\hat{\alpha})$, 22 have $|\hat{\alpha}| > 3 \text{ s.e.}(\hat{\alpha})$, and 29 have $|\hat{\alpha}| > 2 \text{ s.e.}(\hat{\alpha})$.

This analysis suggests, then, that some of the variability apparent in the large observed values of V are a result of assuming $\alpha = 0$ inappropriately, but that the remaining variability is still too large for a Poisson model to be reasonable for all part-base combinations.

CONCLUSION

The Poisson model with parameters λf is not adequate for these failure data. The observed values of V are too large for such a postulated model to be credible. Second, these large values of V cannot be explained reasonably by postulating that part failures follow a Poisson distribution with mean $\alpha + \beta f$ and α not necessarily zero, although the data do give reason to believe that modeling mean failures this way is more appropriate than assuming $\alpha = 0$.

Taken together, these two implications suggest that an appropriate next step in modeling the failure process would be to assume that failures follow a negative binomial distribution with mean $\alpha + \beta f$ and variance-to-mean ratio ρ . If data analysis indicates that $\alpha = 0$ or $\rho = 1$ are reasonable modeling assumptions for some part-base combination, then such a simpler model may be appropriate for them.

One last implication is that for cases where it is reasonable to assume that failures follow a negative binomial distribution with mean αf and variance $\alpha f \rho$, V may not be a desirable estimator for ρ .

III. SUGGESTIONS FOR ESTIMATION

NEGATIVE BINOMIAL WITH MEAN λf

A simple replacement for V that corrects for its bias has not been found. Two natural alternatives have flaws as serious as V 's. Maximum likelihood estimation, however, appears to avoid these but does not yield explicit formulas for the estimates. These three methods will now be examined briefly.

Because $E(V|N) = (R + N)/(R + 1)$ and the UMVUE of p based on N is $(R + N)/R$ in the case where R is known, it may be desirable to correct V by a multiplier of the order of $(R + 1)/R$. One way to do this is to find the estimator \tilde{p} that is a function of V and that, when V is set equal to its conditional expected value, reduces to the desired $(R + N)/R$. This estimator is

$$\tilde{p} = (N - 1)V/(N - V).$$

However, it is easily shown that the upper bound for V for given N is $N(T/f_{\min} - 1)/(n - 1)$, where $f_{\min} = \min \{f_j\}$. This bound is attained when $X_{j^*} = N$ for some j^* satisfying $f_{j^*} = f_{\min}$ and $X_j = 0$ for $j \neq j^*$; and because $T/f_{\min} \geq n$, the bound will be no less than N . Thus \tilde{p} will be infinite or negative with positive probability whatever p , n , λ , or $\{f\}$ are.

The second natural suggestion is to use the method of moments estimators $\tilde{\lambda} = \hat{\lambda} = N/T$ and $\tilde{p} = (\sum X_i^2 - \hat{\lambda}^2 F)/N$, where $F = \sum f_i^2$. However, it can be shown that

$$E(\tilde{p}|N) = (1 - F/T^2)E(V|N) \leq (1 - n)E(V|N)$$

for $N > 0$, and

$$E(\hat{p}) = p\{(1 - p^{-R-1})(1 - F/T^2)R/(R + 1) + p^{-R-1}\} \\ \leq p\{(R + p^{-R-1})/(R + 1) - R(1 - p^{-R-1})/n(R + 1)\} < E(V)$$

for $p > 1$.

Using maximum likelihood estimates can provide an approach to the desideratum of the second paragraph in this subsection. The log likelihood function is

$$\ell(\lambda, p) = R \log p + N \log(1 - p) + \sum \log \binom{k_i + r_i - 1}{k_i}.$$

where $r_i = \lambda f_i / (p - 1)$ and $p = 1/p$. Reparameterize, replacing (λ, p) by (γ, p) , for $\gamma = \lambda / (p - 1)$ and $p = 1/p$, so that $r_i = \gamma f_i$, and momentarily treat γ as fixed at $\hat{\gamma}$. Then

$$\ell(\hat{\gamma}, p) = \hat{\gamma}T \log p + N \log(1 - p) + K(\hat{\gamma}),$$

$K(\hat{\gamma})$ being defined in an obvious way. Differentiating $\ell(\gamma, p)$ with respect to p and solving for \hat{p} yields $\hat{p} = \hat{\gamma}T / (N + \hat{\gamma}T)$, implying that $p = (N + \hat{\gamma}T) / \hat{\gamma}T = (\hat{R} + N) / \hat{R}$ for $\hat{R} = \hat{\gamma}T$. It remains to find $\hat{\gamma}$, which can be done by substituting p into $\ell(\gamma, p)$ and numerically maximizing the resulting equation in γ . By backtransforming and applying standard large-sample maximum likelihood theory to the original log likelihood, regions may be derived for λ and p .

First order bias corrections and second order variance approximations are available for these estimators (see, e.g., Cox and Hinkley, 1974, pp. 309-310); however, they involve very complicated calculations.

NEGATIVE BINOMIAL WITH MEAN $\alpha + \beta f$

The log likelihood for this situation is

$$\ell(p, \alpha, \beta) = R \log p + N \log(1 - p) + \sum \log \binom{k_i + r_i - 1}{k_i}$$

where $r_i = (\alpha + \beta f_i) / (p - 1)$ and $p = 1/p$. If this is reparameterized by setting $\gamma = \alpha / (p - 1)$, $\delta = \beta / (p - 1)$, and $p = 1/p$, the reparameterized log likelihood is

$$\ell(p, \gamma, \delta) = (n\gamma + \delta T) \log p + N \log (1 - p) + K(\gamma, \delta),$$

$K(\gamma, \delta)$ being defined in an obvious way. Temporarily setting $\gamma = \hat{\gamma}$ and $\delta = \hat{\delta}$ and maximizing in p yields

$$\hat{p} = (n\hat{\gamma} + \hat{\delta}T)/(N + n\hat{\gamma} + \hat{\delta}T),$$

so that

$$\hat{p} = (N + n\hat{\gamma} + \hat{\delta}T)/(n\hat{\gamma} + \hat{\delta}T) = (\hat{R} + N)/\hat{R}$$

for $\hat{R} = n\hat{\gamma} + \hat{\delta}T$. Thus, maximum likelihood again provides an approach to the desideratum of the previous subsection. Estimates $\hat{\gamma}$ and $\hat{\delta}$ can be found by substituting \hat{p} into $\ell(p, \gamma, \delta)$ and maximizing numerically in γ and δ . As before, back transformation gives estimates for α and β and approximate confidence regions may be obtained using standard large sample theory.

Computing the method of moments estimator requires the solution of a system of three nonlinear equations and will not be pursued further.

IV. CONCLUSION

The consideration of competing spare parts supply systems requires understanding how well levels of parts failures under wartime conditions can be predicted. The ability to predict levels of parts failures is strongly affected by at least two types of uncertainty: about the numbers of failures that will occur assuming the model is correct, and about the adequacy of the model as an approximation of the process generating parts failures.

The first type of uncertainty is systematically and inherently understated by modeling failures as Poisson random variables. Distributional results derived here make this quite clear. A model that allows more variability, such as a negative binomial model, would be more appropriate. The data also indicate that prevailing beliefs about variation within such negative binomial models err on the optimistic side, and that inherent variability is large, for some parts at least.

The second type of uncertainty can be accommodated partly by using models with more parameters; this is the course taken here by the suggestion that mean failures be modeled as $\alpha + \beta f$ without assuming $\alpha = 0$. Some properties of possible estimators for these models have been examined here also. Although this improvement addresses some of the issues mentioned in Sec. I, it leaves several issues untouched. To study the issues of stationarity of failure distribution and the relationship between flying hours and mean failures, more and better data are needed. In particular, it would be useful to have many years of data to check for seasonality in failures and to observe failure behavior under a broader range of flying hours, especially numbers of hours that would be expected in wartime. Studying the question of independence of disjoint time intervals awaits the elaboration of appropriate probabilistic models that allow for dependence, and of suitable estimation methods for them.

Appendix A

DERIVATION OF $E(V)$ UNDER POISSON ASSUMPTIONS

This appendix derives $E(V|N)$ assuming that X_i is Poisson with parameter μ_i . When $\mu_i = \lambda f_i$, the Poisson assumptions hold and the expectations derived there follow.

First,

$$\begin{aligned} E[(X_i - f_i \hat{\lambda})^2 | N] &= \text{var}(X_i | N) + [E(X_i | N) - f_i \hat{\lambda}]^2 \\ &= Np_i(1 - p_i) + [Np_i - f_i N/T]^2 \\ &= Np_i(1 - p_i) + N^2(p_i - q_i)^2, \end{aligned}$$

where $p_i = \mu_i / \Sigma \mu_i$ and $q_i = f_i / T$, using the fact that $X_i | N$ is binomial with parameters N and p_i . It follows readily that

$$E(V|N) = (n - 1)^{-1} \{ N \Sigma (p_i - q_i)^2 / q_i + \Sigma p_i(1 - p_i) / q_i \}$$

for $N > 0$. Because V was defined to be one for $N = 0$,

$$E(V) = P(N = 0) + \sum_{m=1}^{\infty} E(V|N = m)P(N = m).$$

Because N has a Poisson distribution with parameter $\Lambda = \Sigma \mu_i$,

$$\begin{aligned} E(V) &= e^{-\Lambda} \{ 1 - (n - 1)^{-1} \Sigma (p_i - q_i)^2 / q_i \} \\ &\quad + (n - 1)^{-1} \{ \Lambda \Sigma (p_i - q_i)^2 / q_i + \Sigma p_i(1 - p_i) / q_i \} \\ &= e^{-\Lambda} + (n - 1)^{-1} \{ (\Lambda - e^{-\Lambda}) \Sigma (p_i - q_i)^2 / q_i + \Sigma p_i(1 - p_i) / q_i \}. \end{aligned}$$

Appendix B

DERIVATION OF $\text{VAR}(V/N)$ AND $\text{VAR}(V)$ UNDER POISSON ASSUMPTIONS

This appendix derives $\text{var}(V|N)$ and $\text{var}(V)$ under the assumption that the number of failures in a given period has a Poisson distribution with parameter μ_i . The formula needed for Sec. II may be obtained by substituting $\mu_i = \alpha + \beta f_i$.

Using two facts, that $X_i|N$ is binomial with parameters N and $p_i = \mu_i/\Sigma \mu_j$, and that the joint distribution of $(X_i, X_j, N - X_i - X_j)|N$ is trinomial with index N and probabilities p_i, p_j , and $1 - p_i - p_j$, it is a matter of straightforward algebra to show that

$$\text{var}(V|N) = T^2(n-1)^{-2}\{4NA_1 + A_2/N + A_3\},$$

where

$$A_1 = a_5 - a_2^2, A_2 = 8a_5 - 6a_4 + a_3 + 4a_1a_2 - 6a_2^2 - a_1^2,$$

$$A_3 = -12a_5 + 6a_4 - 4a_1a_2 + 10a_2^2, \text{ and } a_1 = \Sigma p_i/f_i,$$

$$a_2 = \Sigma p_i^2/f_i, a_3 = \Sigma p_i/f_i^2, a_4 = \Sigma p_i^2/f_i^2, \text{ and}$$

$$a_5 = \Sigma p_i^3/f_i^2.$$

The unconditional variance of V is obtained by using

$$\text{var}(V) = \text{var}[E(V|N)] + E[\text{var}(V|N)].$$

Employing the approximation from Sec. II and ignoring the term $e^{-\Lambda}$, where $\Lambda = \Sigma \mu_i$,

$$E(\text{var}(V|N)) = T^2(n-1)^{-2}\{4\Lambda A_1 + A_2/(\Lambda-1) + A_3\}.$$

It is convenient to derive $\text{var}(E(V|N))$ by rearranging terms to avoid difficulties introduced by setting $V = 1$ when $N = 0$. Let

$$g(m) = (n - 1)^{-1} \{ m \sum (p_i - q_i)^2 / q_i + \sum p_i (1 - p_i) / q_i \}$$

where $q_i = f_i / T$. This is $E(V|N = m)$ for $m > 0$. Also, let $G = E(g(N))$. Because N has a Poisson distribution with parameter Λ ,

$$\begin{aligned} \text{var}(E(V|N)) &= \sum_{m=1}^{\infty} (g(m) - E(V))^2 P(N = m) + (1 - E(V))^2 e^{-\Lambda} \\ &= \sum_{m=0}^{\infty} [g(m) - G]^2 P(N = m) + H, \end{aligned}$$

where

$$H = e^{-\Lambda} [(1 - E(V))^2 - \sum p_i (1 - p_i) / q_i - (G - E(V))^2 - (G - E(V))(1 - G)].$$

Thus, $\text{var}(E(V|N)) = (n - 1)^{-2} \Lambda [\sum (p_i - q_i)^2 / q_i]^2 + H$, and for small $e^{-\Lambda}$, H may be ignored.

Appendix C

Table C.1

ESTIMATES AND OTHER QUANTITIES FROM FITTING POISSON
MODELS WITH MEAN A LINEAR FUNCTION OF FLYING HOURS

V	$\hat{\alpha}$	$\hat{\beta}$	se($\hat{\alpha}$)	se($\hat{\beta}$)	corr($\hat{\alpha}, \hat{\beta}$)	E(V)	var(V)	n	P(V > observed V)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
.38	-.726	.001859	12.700	.002512	-.996	1.000	.327	7	.903
.58	-2.851	.006261	7.634	.010245	-.998	1.013	.310	7	.777
.68	-2.619	.002859	2.472	.000674	-.944	1.064	.209	11	.797
.91	1.262	.002112	3.922	.000728	-.969	1.015	.174	13	.560
1.00	-4.812	.010401	5.772	.007488	-.996	1.033	.186	12	.473
1.01	5.595	.001984	4.146	.000906	-.953	1.286	.448	9	.597
1.03	4.782	.005043	11.173	.001776	-.989	1.019	.174	13	.470
1.03	-10.442	.017909	6.303	.008252	-.997	1.165	.222	12	.577
1.10	5.798	.002566	8.668	.001374	-.989	1.044	.184	13	.433
1.22	-1.981	.001461	1.554	.000439	-.944	1.065	.199	11	.330
1.29	-13.809	.005895	8.676	.001431	-.987	1.214	.269	11	.320
1.41	-2.980	.002978	2.072	.000567	-.928	1.085	.195	12	.220
1.45	-.443	.005035	6.859	.008792	-.997	1.000	.177	12	.120
1.45	-10.366	.015683	8.021	.010391	-.999	*	*	12	*
1.50	6.171	-.001602	8.091	.010282	-.997	1.055	.204	12	.123
1.64	6.708	.001392	7.998	.001262	-.991	1.083	.200	13	.107
1.71	25.253	-.022562	11.601	.014614	-.998	1.532	.426	12	.350
1.77	25.365	-.000130	10.243	.001599	-.991	1.679	.452	13	.373
1.81	1.798	.001996	7.533	.001196	-.990	1.009	.170	13	.030
1.83	-4.648	.003519	2.258	.000622	-.938	1.231	.231	12	.100
1.99	-.296	.002393	2.702	.000686	-.949	.997	.179	12	.013
2.02	9.758	-.000477	7.494	.001275	-.991	1.369	.634	7	.190
2.12	-7.822	.003404	6.214	.001008	-.987	1.096	.190	13	.027
2.13	-12.033	.019514	4.442	.005990	-.995	1.244	.239	12	.063
2.16	12.865	-.001006	3.809	.000655	-.981	2.645	1.001	13	.693
2.28	11.663	-.000287	3.947	.000691	-.975	1.991	.633	13	.320
2.38	-4.129	.003868	4.728	.000893	-.972	1.051	.177	13	.003
2.41	18.158	-.019012	10.257	.013223	-.998	1.440	.568	9	.067
2.45	8.537	.003144	12.172	.015515	-.997	1.045	.200	12	.003
2.51	15.206	-.000205	4.546	.000798	-.973	2.186	.708	13	.303
2.54	-7.797	.014424	5.963	.007782	-.996	1.091	.203	12	0
2.68	-8.396	.004056	3.702	.000726	-.967	1.306	.236	13	.007
2.70	14.278	.003035	13.124	.002165	-.991	1.155	.301	10	.013
2.84	1.121	.002219	7.523	.001684	-.986	1.008	.337	7	.010
2.90	-43.958	.061981	6.994	.009586	-.999	*	*	12	*

V	$\hat{\alpha}$	$\hat{\beta}$	se($\hat{\alpha}$)	se($\hat{\beta}$)	corr($\hat{\alpha}, \hat{\beta}$)	E(V)	var(V)	P(V > n observed V)	
								(8)	(9) (10)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
3.14	-5.049	.005797	3.229	.000674	-.939	1.146	.218	12	0
3.15	7.797	-.000577	4.777	.001487	-.981	1.621	.796	7	.057
3.15	7.519	.004659	4.290	.001058	-.941	1.326	.327	12	.007
3.15	5.073	.000641	3.888	.000701	-.978	1.242	.276	13	.003
3.35	9.799	-.000103	3.734	.000656	-.974	1.774	.530	13	.033
3.35	63.208	.000485	23.886	.004534	-.995	1.733	.467	12	.023
3.36	-17.326	.025741	16.076	.020651	-1.000	*	*	12	*
3.38	-6.088	.011391	6.016	.007813	-.997	1.065	.194	12	.007
3.73	-.571	.003320	2.573	.000672	-.927	.997	.179	12	0
3.77	21.233	-.001606	4.920	.000899	-.973	3.310	1.537	10	.350
3.82	36.181	.000756	25.242	.004639	-.996	1.360	.451	9	.007
3.83	-3.454	.003923	2.786	.000765	-.940	1.093	.220	11	0
3.92	-6.254	.004965	2.499	.000700	-.929	1.311	.258	12	0
3.94	8.479	.000011	4.341	.000773	-.982	1.636	.465	13	.010
4.01	-12.277	.005804	12.569	.002423	-.995	1.090	.210	12	0
4.10	-17.779	.033023	12.413	.016036	-.998	1.224	.245	12	0
4.16	-22.025	.006090	7.832	.001547	-.994	1.550	.343	12	0
4.30	29.394	.004525	8.555	.001546	-.971	2.117	.612	13	0
4.42	20.150	.001198	4.923	.001203	-.952	3.144	1.270	11	.137
4.53	-9.884	.005141	11.980	.002308	-.995	1.061	.200	12	0
4.65	-4.682	.014296	5.784	.001496	-.939	1.043	.193	12	0
4.85	14.496	.000176	10.876	.002064	-.995	1.172	.252	12	0
5.07	-124.014	.162206	23.615	.030812	-1.000	*	*	11	*
5.08	15.661	.003134	12.292	.001940	-.991	1.183	.237	13	0
5.24	-10.461	.009528	6.720	.001281	-.967	1.160	.210	13	0
5.26	.325	.003451	4.694	.000877	-.968	1.002	.167	13	0
5.52	-27.993	.008101	9.715	.001909	-.994	1.616	.368	12	0
5.53	-16.540	.015643	16.726	.002678	-.989	1.078	.190	13	0
5.55	14.343	.003541	15.628	.002982	-.994	1.079	.213	12	0
5.57	-14.772	.012369	14.667	.002350	-.989	1.080	.190	13	0
5.63	-6.142	.002678	6.000	.000969	-.989	1.072	.183	13	0
5.78	10.589	.000478	5.208	.000932	-.981	1.628	.445	13	0
6.02	-54.180	.022087	21.404	.004143	-.995	1.554	.368	12	0
6.31	-50.185	.012110	13.447	.002630	-.997	3.186	.708	12	0
6.34	-29.529	.008996	7.599	.001265	-.984	1.762	.350	13	0
6.35	27.142	-.020186	12.328	.015555	-.997	1.411	.362	12	0
6.99	-54.733	.012882	11.654	.002304	-.997	*	*	12	*
7.09	-10.435	.005586	8.874	.001430	-.988	1.098	.193	13	0
7.28	48.772	-.002856	7.248	.001249	-.976	5.996	2.487	13	.223
7.38	-104.058	.026808	11.370	.001933	-.985	5.053	.984	13	.013
8.11	66.438	-.075362	16.322	.020428	-.999	*	*	12	*
8.25	50.765	-.033217	19.743	.025002	-.997	1.651	.457	12	0
8.73	-35.961	.052196	13.257	.017233	-.999	*	*	12	*
10.36	31.387	-.000279	6.196	.001084	-.970	3.313	1.186	13	0

V	$\hat{\alpha}$	$\hat{\beta}$	se($\hat{\alpha}$)	se($\hat{\beta}$)	corr($\hat{\alpha}, \hat{\beta}$)	E(V)	var(V)	n	P(V > observed V)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
11.11	-48.929	.013809	13.114	.002573	-.995	2.182	.539	12	0
11.22	-24.905	.015779	6.886	.001355	-.957	1.650	.347	13	0
11.34	-27.346	.030189	7.181	.001926	-.935	2.006	.496	12	0
13.02	-21.252	.013812	3.745	.001090	-.930	2.628	.607	12	0
14.09	-55.819	.015297	9.574	.001935	-.991	2.484	.621	12	0
14.27	-33.828	.019319	21.990	.004241	-.994	1.195	.248	12	0
15.05	120.416	-.008942	18.086	.002791	-.993	6.726	2.782	13	C
15.25	38.252	-.002150	13.236	.002063	-.995	2.519	.845	13	0
16.29	43.909	-.001820	5.952	.001340	-.957	8.441	3.796	12	0
17.94	102.716	-.008528	16.106	.002479	-.993	6.464	2.729	13	0
18.00	-108.728	.027879	12.591	.002556	-.993	4.703	1.199	12	0
18.64	-.942	.011896	5.819	.001482	-.945	1.000	.181	12	0
19.35	-39.378	.010669	9.807	.001945	-.994	2.077	.496	12	0
19.84	67.724	-.000254	7.823	.001789	-.954	10.582	4.690	12	0
21.06	83.637	-.008222	7.500	.001233	-.976	13.579	6.392	13	0
22.44	-38.009	.024310	29.261	.005624	-.996	1.187	.246	12	0
22.46	60.663	-.006681	5.570	.000878	-.976	11.678	5.604	13	0
47.63	-14.467	.013260	21.675	.004163	-.995	1.042	.196	12	0
50.70	653.877	-.054330	38.079	.005842	-.992	35.573	16.230	13	0
57.41	-29.283	.051312	12.104	.003102	-.953	1.617	.385	12	0
95.31	-505.081	.131377	37.952	.007482	-.995	17.436	4.798	12	0

* indicates that for that part and base, $\hat{\alpha} + \hat{\beta}f_j < 0$
for some j, making the formulas for E(V) and var(V) inappropriate.
Cols. (2) and (3). The estimates $\hat{\alpha}$ and $\hat{\beta}$ were obtained
by maximizing the log likelihood numerically.
Cols. (4) and (5). The approximate errors were obtained from the
inverse of the observed information matrix.
Col. (6) is the approximate correlation between $\hat{\alpha}$ and $\hat{\beta}$.
Cols. (7) and (8). E(V) and var(V) are computed using $\hat{\alpha}$ and $\hat{\beta}$.
Col. (9). n = the number of observations for that part and base.
Col. (10) is the proportion of 300 simulated values of V greater than
the observed value of V, where the pseudo observations were generated
from a Poisson distribution with parameter $\hat{\alpha} + \hat{\beta}f_j$.

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